Inference of Bidders' Risk Attitudes in Ascending Auctions with Endogenous Entry*

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Abstract

Bidders' risk attitudes have key implications for the choices of revenue-maximizing auction formats. In ascending auctions, bid distributions do not provide information about risk preference. We infer risk attitudes using distributions of transaction prices and participation decisions in ascending auctions with entry costs. Nonparametric tests are proposed for two distinct scenarios: first, the expected entry cost can be consistently estimated from the data; second, the data does not report entry costs but contains exogenous variation in potential competition and auction characteristics. We also show the identification of risk attitudes in ascending auctions with selective entry, where bidders receive entry-stage signals correlated with their values.

Keywords: Ascending auctions, risk attitudes, endogenous entry, tests

JEL Classification Codes: D44, C12, C14

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1 Introduction

We propose nonparametric tests to infer bidders' risk attitudes in ascending (or open out-cry) auctions with endogenous entry. In these auctions, potential bidders observe some entry costs, e.g., bid preparation/submission costs and/or information acquisition costs, that they need to incur before learning private values, and decide whether to pay the costs to be active in the bidding stage. Bidders make rational entry decisions by comparing expected utility from entry with that from staying out, based on their knowledge of entry costs or preliminary signals of private values to be realized in the subsequent bidding stage.

Inference of bidders' risk attitudes has important implications for sellers' choices of the revenue-maximizing auction format. When participation of bidders is exogenously given and fixed, the Revenue Equivalence Theorem states that expected revenues from first-price and ascending auctions are the same if bidders are risk-neutral in an environment with symmetric independent private values (IPV). If bidders are risk-averse, however, Matthews (1987) showed that in such environments first-price auctions yield higher expected revenues than ascending auctions. Bidders' risk attitudes also affect revenue rankings among symmetric IPV auctions when participation decisions are endogenous. For risk-neutral bidders, Levin and Smith (1994) showed that any given entry cost induces the same entry probabilities in first-price auctions (with each entrant observing the number of the other entrants) and in ascending auctions. Thus the Revenue Equivalence Theorem implies expected revenues must be the same from both first-price and ascending formats under endogenous entry. On the other hand, Smith and Levin (1996) established the revenue ranking of first-price over ascending auctions under endogenous entry for risk-averse bidders, except for the case with decreasing absolute risk aversions (DARA).[1]

While some earlier papers had studied the identification and estimation of bidders' risk attitudes in *first-price* auctions (e.g. Bajari and Hortascu (2005), Campo, Guerre, Perrigne and Vuong (2011) and Guerre, Perrigne and Vuong (2009)), inference of risk attitudes in ascending auctions remains an open question. Athey and Haile (2007) point out that bidders' risk attitudes cannot be identified from bids *alone* in ascending auctions where participation is given exogenously. This is because bidding one's true value is a weakly dominant strategy in ascending auctions, *regardless of bidders' risk attitudes*. Thus, bidders with various risk attitudes could generate the same distribution of bids in Bayesian Nash equilibria. The

[1] Even in the case with DARA, first-price auctions yield higher expected revenues than ascending auctions when entry costs are low enough. To see this, consider a simple case where entry costs are low enough so that the difference between entry probabilities in first-price and ascending auctions are sufficiently small. In such a case, these two probabilities are both close to 1 and only differ by some small $\varepsilon > 0$. By Matthews (1987), conditioning on any given number of entrants, ascending auctions have smaller expected revenues than first-price auctions. When the difference between the two entry probabilities is small enough, such a revenue ranking result is preserved.

distribution of bids from entrants alone is therefore not sufficient for inferring bidders' risk attitudes.

In this paper, we propose tests for bidders' risk attitudes based on transaction prices *and* entry decisions under two empirically relevant data scenarios. In the first scenario, we assume that the data has information that allows researchers to consistently estimate the expectation of entry costs; and in the second scenario, we assume that the data does not provide any information about the level of entry costs, but does contain variation in the number of potential bidders and some auction-level heterogeneity. In both cases, we show how to relate the distribution of transaction prices and entry decisions to the underlying risk attitudes nonparametrically.

In the first scenario, considered in Section 4, we require that the data contains some noisy measures of entry costs so that the mean of entry costs in the data-generating process can be consistently estimated. This is motivated by the fact that entry costs are often measurable (at least up to some noise) in applications. Examples of entry costs include bid preparation costs (e.g., mailing costs), admission fees or other information acquisition expenses, which are often reported in the data with noise.

The main insight for our test in this scenario can be illustrated using the mixed-strategy entry model (which is analogous to that considered in Levin and Smith (1994) for first-price auctions with risk-neutral bidders). In the entry stage, all potential bidders observe a common entry cost and decide whether to pay the cost and enter an ascending auction in the bidding stage. In a mixed strategy Nash equilibrium in the entry stage, potential bidders' participation in the auction will be in mixed strategies with the mixing probability determined to ensure that a bidder's *ex ante* expected *utility* from entry equals that from staying out. Hence bidders' risk attitudes can be identified by comparing the expected *profits* from entry with its certainty equivalent. As long as the expectation of entry costs can be identified from the data, the distribution of transaction prices and entry decisions alone can be used to make such a comparison.

We apply the analog principle to construct a non-parametric test statistic, using observations of transaction prices and entry decisions as well as estimates of the mean of entry costs. We characterize the limiting distribution of this statistic, and propose a bootstrap test that attains correct asymptotic level and is consistent under any fixed alternative of risk-aversion or risk-loving.

In the second scenario, considered in Section 5, we assume that the data does not provide information about the level of entry costs, but does contain some auction-level heterogeneity (such as some characteristics of the auctioned object) and variation in the numbers of potential bidders. We propose a nonparametric test for risk-aversion for this case, under the assumption that variation in potential competition is exogenous (in the sense that it does not alter the marginal distribution of private values once conditional on the observed auction features). Refraining from parametric restrictions on how observed auction features change the distribution of private values, we formalize how risk attitudes determine entry probabilities under various auction features and potential competition. Our test is based on the idea that the curvature of utility functions affects how the ratios between the changes in bidders' interim utilities compare with the ratios between the changes in expected private values, under different pairs of observed auction features. The main finding is that, even if entry costs are unreported in data, we could relate these two ratios to the observed distributions of transaction prices and entry decisions, by exploiting the variation in auction features and the number of potential bidders. We provide encouraging Monte Carlo evidence for the finite-sample performance of our test.

In Section 6, we discuss possible extensions of our tests by removing two of the key assumptions. First, we show that when bidders' values are affiliated, it is possible to derive testable implications for risk attitudes using existing results on the sharp bounds for the surplus of risk-neutral bidders in ascending auctions (Aradillas-Lopez, Gandhi and Quint (2013)). Second, we show that if entry is selective (e.g., when potential bidders observe signals correlated with private values to be drawn in the bidding stage), then the idea of testing risk attitudes through the identification of risk-premium applies, provided the data contains continuous variation in observed entry costs.

It is worth noting that by "inference of risk attitudes" we mean to make a data-supported conclusion about whether bidders' utility functions are concave, linear or convex. We do not address the question of how to recover the utility function completely over its domain, which is left for future research.

The remainder of the paper is structured as follows. In Section 2 we discuss the related literature; in Section 3 we present the model of ascending auctions with endogenous entry; in Sections 4 and 5 we describe the theoretical results for our tests under two data scenarios, and discuss the inference using proposed test statistics; in Section 6 we discuss how to extend our test to auctions with selective entry or affiliated private values; and in Section 7 we conclude. Proofs are collected in the appendix.

2 Related Literature

This paper contributes to two branches of the literature on structural analyses of auction data. The first branch includes papers that analyze the equilibrium and its empirical implications in auctions with endogenous entry and *risk-neutral* bidders. These include Levin and Smith (1994), Li (2005), Ye (2007) and Li and Zheng (2009). Marmer, Shneyerov and Xu (2011) study a model of first-price auctions with risk-neutral bidders and selective entry, and discuss testable implications of various nested entry models. Roberts and Sweeting (2010) estimate a model of ascending auctions with selective entry and risk-neutral bidders. Gentry and Li (2013) provide partial identification results for ascending auctions with risk-neutral bidders when entry is selective. They derive sharp bounds on the distribution of private values conditional on signals, using variation in factors that affect bidders' entry behaviors (such as the number of potential bidders and entry costs). They also apply these results to bound counterfactual seller revenues under alternative auction rules. Aradillas-Lopez, Gandhi and Quint (2013) provide partial identification results for ascending auctions where bidders' private values are affiliated, exploiting exogenous variation in the number of entrants, or active bidders.

The second branch includes papers that study the identification and estimation of bidders' utility functions and the distribution of private values in first-price auctions without endogenous entry. Campo, Guerre, Perrigne and Vuong (2011) show how to estimate a semiparametric model of *first*-price auctions with risk-averse bidders when the identification of a parametric utility function is assumed. Bajari and Hortascu (2005) use exogenous variation in the number of bidders in first-price auctions to semi-parametrically estimate the utility function while leaving the distribution of bidders' private values unrestricted. Guerre, Perrigne and Vuong (2009) use exogenous variation in the number of potential bidders to non-parametrically identify bidders' utility functions along with the distribution of private values in first-price auctions. Lu and Perrigne (2008) consider a context where data contain bids from both first-price and ascending auctions that involve bidders with the same utility function and the same distribution of private values. They first use bids from ascending auctions to recover the distribution of private values, and then use bids from first-price auctions to recover the utility function.

Our work in this paper contributes to these two branches of empirical auction literature by studying a model which endogenizes bidders' entry decisions and relaxes the risk-neutrality assumption at the same time. To the best of our knowledge, our paper marks the first effort to non-parametrically infer bidders' risk attitudes in ascending auctions with endogenous entry. Smith and Levin (1996) present some results on the ranking of auction formats in terms of seller revenues when auctions are known to involve risk-averse bidders who make endogenous entry decisions. Their focus is not on the identification of bidders' risk attitudes.

Ackerberg, Hirano and Shahriar (2011) study a class of e-Bay auctions where a typical online ascending auction is combined with an option of paying the buy-out price posted by the seller in order to immediately purchase the object. They show how to identify the bidders' utility functions and the distribution of private values using exogenous variation in the buy-out prices and other auction characteristics. The format of auctions they study is qualitatively different from the one we consider in this paper. We do not embark on a full identification of the utility function in this paper, and therefore require fewer sources of exogenous variation to perform the test. (With observed and exogenous variation in entry costs, identification of the utility function may be possible in our model as well.) Our approach does not rely on variation in entry costs, thus our test can be performed for any given level of entry costs. We also propose a robust method of inference.

Our paper also fits in a category of empirical auction literature on nonparametric tests of the empirical implications/predictions of auction theory. Earlier works in this category include tests of bidders' rationality in first-price auctions with common values in Hendricks, Pinkse and Porter (2003), assessment of winner's curse in first-price auctions with common values in Hong and Shum (2002), tests for the interdependence between bidders' values in Haile, Hong and Shum (2004), and tests for the affiliation between bidders' private values in Li and Zhang (2010) and Jun, Pinkse and Wan (2010).[2] The test proposed by Li and Zhang (2010) uses bidders' entry decisions for testing the affiliation between bidders' private values.

Li, Lu and Zhao (2012) study first-price and ascending auctions with risk-averse bidders and selective entry. They find the ranking between bidders' expected utility under these two auction formats depend on the specific form of risk aversion (DARA, CARA or IARA). Consequently, the ranking of entry probabilities across these two auction formats must also depend on the form of risk aversion. Based on this observation, they propose an original test for inferring the form of bidders' risk aversion using entry behaviors, using data from both first-price and ascending auctions at the same time. Our research focus in this paper differs from theirs in that we study the inference of risk attitudes (with hypotheses being risk-neutral, risk-taking or risk-aversion). Since we do not aim at inferring the form of risk-aversion, our test does not require observation of data from multiple competing auction formats.

3 Ascending Auctions with Endogenous Entry

Consider an empirical context where researchers observe a large number of independent single-unit ascending auctions in the data. Each of these auctions involve N potential bidders who have symmetric independent private values and make endogenous entry decisions. The entry model we consider here is similar to those used for first-price auctions with risk-neutral bidders in Levin and Smith (1994), Li and Zheng (2009) and Marmer, Shneyerov and Xu (2011). In the entry stage, each potential bidder decides whether to incur an entry cost K to become active in the bidding stage. The entry cost is common knowledge among all potential bidders. There is a reserve price r that is known to all potential bidders in the entry stage. Following their entry, each entrant i draws a private value V_i, and competes in an ascending auction in the bidding stage. Across auctions, private values and entry costs are independent draws from distributions $F_{(V_1,.,V_N)|K}$ and F_K respectively, which are common knowledge among all potential bidders prior to entry decisions. Upon entry, each

[2] See Athey and Haile (2007) and Hendricks and Porter (2007) for recent surveys.

entrant may or may not be aware of the total number of entrants (which we denote by A). All bidders in the data share the same bounded von Neumann-Morgenstein utility function $u : \mathbb{R}_+ \to \mathbb{R}$ with $u' > 0$ and the sign of u'' does not change over $\mathbb{R}_+$. A winner who has a private value V_i and pays a price P_i receives a utility of $u(V_i - P_i - K)$.

With a slight abuse of notation, we use N and A to denote the number as well as the set of potential bidders and entrants respectively. Let F_{ξ_1,ξ_2}, $F_{\xi_2|\xi_1}$ denote respectively the joint and conditional distributions of generic random vectors (ξ_1, ξ_2). We use upper cases to denote random variables and lower cases to denote their realizations. We take N as given and fixed in this section and, to simplify exposition, we drop N in the notation when there is no ambiguity.

Assumption 1 *Conditional on K, private values V_i are independent draws from the same continuous marginal distribution $F_{V|K}$ which has positive density almost everywhere with respect to the Lebesgue measure on the support $[\underline{v}, \overline{v}]$. Entry costs across auctions are independent draws from a continuous distribution F_K with a support $[\underline{k}, \overline{k}]$.*

This assumption states that bidders have symmetric, independent private values in the sense that $\Pr(V_1 \leq v_1, ., V_N \leq v_N \mid K = k) = \prod_{i \in N} F_{V|k}(v_i)$ for all k and $(v_1, ., v_N)$. Each entrant i in a bidding stage follows the weakly dominant equilibrium strategy to drop out at his true value V_i if $A \geq 2$. When $A = 1$ in the bidding stage, the lone entrant wins and pays the reserve price r.

Let A_{-i} denote the set (and the number) of entrants that i competes with if he enters. Let the reserve price be binding (i.e. $r > \underline{v}$). Define $P_i \equiv \max\{r, \max_{j \in A_{-i}}\{V_j\}\}$ as i's payment if he enters and wins while all competitors in A_{-i} follow their weakly dominant bidding strategies. Then i's (random) profit in the weakly dominant strategy equilibrium is $(V_i - P_i)_+ - K$, where $(\cdot)_+ \equiv \max\{\cdot, 0\}$. Let $\omega(k; \boldsymbol{\lambda}_{-i})$ denote the expected utility for bidder i conditional on paying an entry cost k and potential competitors entering with probabilities $\boldsymbol{\lambda}_{-i} = (\lambda_j)_{j \in N \setminus \{i\}} \equiv (\lambda_1, ., \lambda_{i-1}, \lambda_{i+1}, ., \lambda_N)$. Under Assumption 1,

$$\omega(k; \boldsymbol{\lambda}_{-i}) \equiv u(-k) F_{V|k}(r) + \int_r^{\overline{v}} h(v, k; \boldsymbol{\lambda}_{-i}) dF_{V|k}(v), \tag{1}$$

where for all $v > r$,

$$\begin{aligned} h(v, k; \boldsymbol{\lambda}_{-i}) \equiv \; & u(v - r - k) F_{P_i}(r \mid k, \boldsymbol{\lambda}_{-i}) + \int_r^v u(v - p - k) dF_{P_i}(p \mid k, \boldsymbol{\lambda}_{-i}) \\ & + u(-k)[1 - F_{Pi}(v \mid k, \boldsymbol{\lambda}_{-i})] \end{aligned} \tag{2}$$

with $F_{P_i}(\cdot|k, \boldsymbol{\lambda}_{-i})$ being the distribution of P_i when $K = k$, and i's potential competitors enter with probabilities $\boldsymbol{\lambda}_{-i}$. Due to the symmetry in the distribution of private values across bidders, $F_{P_i}(.|k, \boldsymbol{\lambda}_{-i})$ does not change with the bidder identity i, and therefore ω does not depend on the bidder identity i. The following lemma characterizes the symmetric mixed strategy equilibrium in the entry stage:

Lemma 1 *Suppose Assumption 1 holds. For any entry cost k with $\omega(k;(1,.,1)) < u(0) < \omega(k;(0,.,0))$, there exists a unique symmetric mixed strategy equilibrium in which all bidders enter with probability λ_k^*, where λ_k^* solves $\omega(k;(\lambda_k^*,.,\lambda_k^*)) = u(0)$.*

Whenever $\omega(k;(0,.,0)) \leq u(0)$ (respectively, $\omega(k;(1,.,1)) \geq u(0)$), the equilibrium entry probabilities are degenerate at 0 (respectively, 1). Thus the condition that $\omega(k;(1,.,1)) < u(0) < \omega(k;(0,.,0))$ can be tested, as long as entry decisions are reported in data. The equilibrium entry process is non-selective, as potential bidders' entry decisions are not based on any entry-stage information that is correlated with the private values to be drawn in the bidding stage.

It is worth noting that we can generalize the model to incorporate auction heterogeneity Z that is known to all potential bidders *and* reported in the data. To do this, we need to condition the private value distribution on Z as well. The characterization of entry and bidding strategies in equilibrium can be derived for any given z using an argument similar to that leading to Lemma 1. In fact, auction heterogeneity reported in the data provides additional identifying power for inferring bidders' risk attitudes when entry costs are not reported in data (see Section 5.)

We assume researchers know the number of potential bidders in auctions. This means either the data reports the number of potential bidders directly, or researchers can construct a measure of N from the data based on institutional details in the specific environments. Such an assumption holds in a variety of applications considered in the literature. For example, Li and Zheng (2009) consider highway mowing auctions by Texas Department of Transportation. In that case, the number of potential bidders for each auction is reported as the observed number of contractors who have requested the official bidding proposal for the project. Similarly, in Krasnokutskaya and Seim (2011), potential bidders in highway procurement auctions in California are defined as companies that have purchased the detailed project plans from California Department of Transportation. Such purchases are recorded for all projects in data. Athey, Levin and Seira (2011) study US Forest Service (USFS) timber auctions. They measure the number of loggers potentially interested in an auction by the number of logging companies that had entered auctions in the same geographic area in the year before. Such a proxy is constructed by combining the data from timber auctions with the US Census data.

4 Inference of Risk Attitudes Using Entry Costs

We now introduce a test for bidders' risk attitudes when the data allows researchers to construct a consistent estimator for the expectation of entry costs. In some contexts, a fixed amount of entry costs, sometimes also known as admission fees, is charged to all entrants and recorded in data. Such admission fees occur in contexts such as: auctions of used or new cars in the U.S. and U.K.; auctions of vintage wines in the U.S.; and auctions of art items, etc. In other applications, entry costs are not admission fees but nevertheless are measurable, at least up to random noise through additional surveys.[3] That said, it is also likely that these noisy measures may systematically underestimate true costs, and taking the average of these noisy measures would lead to a biased estimator for the actual mean. Thus the test based on the sign of risk-premia would be biased towards the direction of risk-averse alternatives.

In addition to its empirical relevance, there is also a theoretical motivation for this assumption. Without variation in potential competition or information about entry costs, bidders' risk attitudes can not be inferred from entry decisions and transaction prices alone without imposing additional parametric assumptions on the structure. To fix ideas, consider a simplified model where all auctions in data share the same fixed entry cost k that is not reported in data. The private value distribution is identified from the distributions of transaction prices and the number of entrants using a standard argument using order statistics; and the equilibrium entry probability is directly recovered from entry decisions. Nonetheless, $u(\cdot)$ and the unknown entry cost k cannot be jointly identified in this case.

To see this, suppose bidders are risk-neutral with a continuous utility function $u(\cdot)$, entry cost k, and entry probability $\lambda^* \in (0,1)$. Fix a continuous private value distribution and let it be independent of entry costs. Now consider a slightly concave utility $\tilde{u}(\cdot) \neq u(\cdot)$ so that $E[\tilde{u}((V_i - P_i)_+ - k) | \lambda^*] > \tilde{u}(0)$. Because the left-hand side is continuous and strictly monotone in k, we can increase k to $\tilde{k} > k$ so that the indifference condition is restored for $(\tilde{u}(\cdot), \tilde{k}) \neq (u(\cdot), k)$. That is, the equilibrium entry probability observed, λ^*, can be rationalized by more than one data-degenerating process. Thus, without variation in potential competition or auction heterogeneity, inference of risk attitudes in English auctions with endogenous entry must utilize at least some partial knowledge of entry costs.

All in all, we acknowledge the assumption that the mean of entry costs can be consistently estimated from the data is strong. In addition to empirical and theoretical motivations above, we also hope to motivate the test under such an assumption as a useful benchmark that shows

[3]For example, in USFS timber auctions considered in Athey, Levin and Seira (2011), entry costs for potential bidders (i.e., millers and loggers located in adjacent geographic regions) consist largely of information acquisition costs. These costs are incurred while performing "cruises" over the auctioned tracts to learn the distribution of diameters and heights of trees, etc. Such private cruises are standard practices institutionalized in the industry, and their costs vary little across millers and loggers. Thus cruise costs can be treated as practically identical for all potential bidders. Besides, it is plausible that researchers can construct a consistent estimator of average costs for cruising tracts with given characteristics, because cruise costs are likely to be measured up to random errors through additional field work of data collection (such as surveys or industry interviews). As cruise costs are orthogonal to bidders' private values (or realized profitability) upon entry, one could reasonably expect there to be no incentive to systematically under- or over-state cruise costs in surveys. In this case, survey data, if collected, could well be expected to provide a consistent estimator for expected entry costs.

how entry behaviors can be informative about bidders' risk premia, which in turn is useful for inferring their risk attitudes.

4.1 Identifying Bidders' Risk Attitudes

Assume that the number of potential bidders is fixed and known to researchers. Our test for risk attitudes in this subsection builds on the simple intuition that the certainty equivalent for *risk-averse* bidders is strictly less than ex ante profits from entry. To fix ideas, we first show the difference between these two quantities can be recovered from the distribution of entry decisions and transaction prices when entry costs are fixed and reported in data. Later in this subsection, we show how to extend this approach under a more practical scenario where the data does not report entry costs perfectly, but allows researchers to construct a consistent estimator for its mean.

First, we show that bidders' *ex ante* expected profits are identified. Let λ_k^* denote entry probabilities in the symmetric BNE when the common entry cost is k. Let $\pi(k)$ denote the ex ante profit for a bidder i if he enters, conditional on entry cost k and that each of his potential competitors enters with probability λ_k^*. That is, $\pi(k) \equiv E[(V_i - P_i)_+ - K \mid K = k]$. (Strictly speaking, the definition of $\pi(k)$ is conditional on the event that "$\lambda_j = \lambda_k^* \ \forall j \neq i$". We suppress this from the notation for simplicity.) Note $\pi(\cdot)$ is independent of bidder identities due to the symmetry in private value distributions. For all $s \leq m$, let $V^{(s:m)}$ denote the s-th smallest among m independent draws of private values from a parent distribution $F_{V|k}$. Let $F_{V^{(s:m)}|k}$ denote the distribution of this order statistic given entry cost k.

Proposition 1 *Suppose Assumption 1 holds and entry costs are reported in data. For any k with $0 < \lambda_k^* < 1$, $\pi(k)$ is identified from bidders' entry decisions and the distribution of transaction prices.*

The intuition builds on two simple observations: First, Assumption 1 guarantees $F_{V|k}$ can be recovered from the distribution of transaction prices under cost k. This is because, with the number of entrants A reported in data, the distribution of prices is a distribution of the second-largest order statistics out of a known number of i.i.d. draws from $F_{V|k}$. Second, once $F_{V|k}$ is recovered, $\pi(k)$ can be calculated as a known functional of the distribution of private values. It is worth noting that identification of ex ante surplus $E[(V_i - P_i)_+ \mid K = k]$ *per se* does not rely on assumed knowledge of entry costs from data. Rather, it could be recovered for an unknown entry cost k, as long as researchers are aware of which auctions in data share this particular level of entry cost k.

Proposition 2 *Suppose Assumption 1 holds. For any k such that $0 < \lambda_k^* < 1$, $\pi(k) = 0$ if and only if bidders are risk-neutral, and $\pi(k) > 0$ (or < 0) if and only if bidders are risk-averse (or risk-loving).*

Proof. Lemma 1 showed that for any such k and in a symmetric BNE, bidders enter with probability λ_k^*, where $\omega(k;\lambda_k^*) \equiv E[u((V_i - P_i)_+ - k) \mid K = k,$ "$\lambda_j = \lambda_k^* \; \forall j \neq i$"$] = u(0)$. Thus, zero is the certainty equivalent associated with $u(\cdot)$ and the distribution of $(V_i - P_i)_+$ given an entry cost k and the value distribution $F_{V|k}$. It then follows that $\pi(k) > 0$ if $u'' < 0$ (bidders are risk-averse). Likewise, $\pi(k) = 0$ (or $\pi(k) < 0$) if bidders are risk-neutral (or risk-loving). □

If the exit prices by all losing bidders are reported in data, the distribution of other order statistics $V^{(\tilde{m}:m)}$ with $\tilde{m} \leq m-2$ provides a source of over-identification of $\pi(k)$. This is because the one-to-one mappings between $F_{V|k}$ and $F_{V^{(\tilde{m}:m)}|k}$ exists for $\tilde{m} \leq m-2$ as well. Such over-identification should be exploited to improve efficiency in the estimation.We now turn to the more practical scenario where the data provide enough information to identify the *expectation* of entry costs, even though the exact entry cost in each auction is not reported. Such a scenario is relevant, for example, when the data do not report K but provide noisy measures of costs $\tilde{K} = K + \epsilon$ with $E(\epsilon) = 0$. In such cases, $\mu_K \equiv E(K) = E(\tilde{K})$ is identified.

Corollary 1 (*of Proposition 2.*) *Suppose Assumption 1 holds. (a) If $0 < \lambda_k^* < 1$ for all $k \in [\underline{k}, \overline{k}]$, then $E[\pi(K)] = 0$ when bidders are risk-neutral, and $E[\pi(K)] > 0$ (or < 0) when bidders are risk-averse (or risk-loving); (b) If K is independent of $(V_i)_{i \in N}$ and $E(K)$ is identified, then $E[\pi(K)]$ is identified from entry decisions and the distribution of transaction prices.*

The key idea underlying this result is that, with the sign of $u''(.)$ assumed fixed over its domain, the testable implications in Proposition 2 are preserved after integrating out entry costs under conditions of Corollary 1. Thus, with the expectation of entry costs assumed known in Corollary 1, the test only requires researchers to recover $E[\pi(K)]$ (i.e., *ex ante* profits after K is integrated out). The latter is achieved by exploiting two facts. First, once conditioning on the number of entrants, bidders' expectation of their surplus $(V_i - P_i)_+$ is independent of entry costs. This is because of the orthogonality condition between private values and the entry costs assumed in Part (b) of Corollary 1. Thus the expectation of $(V_i - P_i)_+$ given the number of competing entrants A_{-i} can be recovered from the data as in Proposition 2. Second, the properties of the (binomial) distribution of the number of entrants can be used to relate the unconditional distribution of the number of competitors for i in the bidding stage to observed distributions.

Our approach here can be extended to allow for observed auction-level heterogeneity in the data. To do so, we need to modify Proposition 1 and Proposition 2 and Corollary 1 by conditioning the assumptions and results therein on the observable auction-level heterogeneity.

The main limitation of the approach in this subsection is of course its reliance on the existence of a consistent estimator for the mean of entry costs. This assumption does not hold when the noisy measures of entry costs in the data are subject to systematic omissions or overstatement. In those cases, the average of these noisy measures yields a biased estimator for expected entry costs.[4] In Section 5 we propose an alternative test that does not require knowledge of entry costs and thus is not subject to this limitation. This is done by exploiting exogenous variation in potential competition and auction heterogeneity reported in data.

4.2 Test Statistic and Bootstrap Inference

We propose a statistic for testing the null hypothesis that bidders are risk-neutral under the conditions of Corollary 1. The entry costs K vary across auctions independently from $(V_i)_{i\in N}$; and for each auction the researcher only observes a noisy measure of the entry cost $\tilde{K} = K + \epsilon$, where ϵ has zero mean and is independent of K and $(V_i)_{i\in N}$. The zero mean of ϵ ensures the sample mean of $\tilde{K}$ is an unbiased and consistent estimator for $E(K)$. To rule out uninteresting cases with degenerate entry decisions, we maintain that $0 < \lambda_k^* < 1$ for all $k \in [\underline{k}, \overline{k}]$ throughout the rest of this section.[5] Let r denote a binding reserve price ($F_V(r) > 0$) that is fixed in the data and known to all potential bidders. To simplify exposition, fix N and suppress it from the notation throughout this section and Appendix B.

Our goal is to infer which of the three hypotheses below is best supported by data:

$$\begin{aligned} H_0 &: \tau_0 = 0 \text{ (risk-neutral);} \\ H_A &: \tau_0 > 0 \text{ (risk-averse); and } H_L : \tau_0 < 0 \text{ (risk-loving).} \end{aligned}$$

where $\tau_0 \equiv E[\pi(K)]$. The data contain T independent auctions, each of which is indexed by t and involves N potential bidders. Let A_t denote the number of entrants in auction t. Let W_t define the transaction price in auction t. If the object is sold, then $W_t = \max\{r, V^{(A_t-1:A_t)}\}$ when $A_t \geq 2$ and $W_t = r$ when $A_t = 1$. Otherwise, define $W_t < r$.

Our test statistic $\hat{\tau}_T$ is a multi-step estimator for τ_0 based on the sample analog principle. First, for $m \geq 2$, estimate the distribution of transaction prices:

$$\hat{F}_{W|A=m,T}(s) \equiv \tfrac{1}{T}\textstyle\sum_{t\leq T} 1\{W_t \leq s,\ A_t = m\}/\tfrac{1}{T}\textstyle\sum_{t\leq T} 1\{A_t = m\}$$

for any $r \leq s \leq \overline{v}$. Then for any $m \geq 2$, estimate F_V by:

$$\hat{F}_{V,T}(s) \equiv \tfrac{1}{N-1}\textstyle\sum_{m=2}^{N} \phi_m^{-1}(\hat{F}_{W|A=m,T}(s)) \text{ for } s \geq r, \tag{3}$$

[4] If the direction of bias is known to be upward then averaging these noisy measures gives a consistent estimator for some upper bound on expected entry costs. One can still test the null that *ex ante* surplus $E[(V_i - P_i)_+ \mid K = k]$ does not exceed the upper bound of expected entry costs. If the null is rejected then we can conclude bidders are risk averse. Nonetheless such a test has limited value in practice, for it is inconclusive about risk attitudes when the null is not rejected.

[5] While Corollary 1 allows for correlation between ϵ and K, we maintain independence between them throughout this section only for the sake of simplifying the derivation of the limisting distribution of the test statistic.

where $\phi_m(t) \equiv t^m + mt^{m-1}(1-t)$ so that its inverse ϕ_m^{-1} maps from the distribution of a second-largest order statistic from m independent draws to the parent distribution.

Note this definition of $\hat{F}_{V,T}(s)$ takes advantage of the over-identifying power of Assumption 1 by taking an average of the $N-1$ independent estimators for F_V. Our test would remain consistent against any fixed alternative and asymptotically valid under the null even with the average in (3) replaced by any one of the $N-1$ estimates in $\{\phi_m^{-1}(\hat{F}_{W|A=m,T}(.)) : 2 \leq m \leq N\}$.

Next, define $\zeta_a \equiv E[(V_i - P_i)_+ \mid A_{-i} = a]$ for $0 \leq a \leq N-1$. Estimate ζ_a by:

$$\hat{\zeta}_{a,T} \equiv \int_r^{\overline{v}} \left[\hat{F}_{V,T}(s)\right]^a \left[1 - \hat{F}_{V,T}(s)\right] ds, \tag{4}$$

where the integral can be calculated using mid-point approximations. Estimate the distribution of A_{-i} by the column vector $\hat{\rho}_T \equiv [\hat{\rho}_{0,T}, \hat{\rho}_{1,T}, .., \hat{\rho}_{N-1,T}]'$ where:

$$\hat{\rho}_{a,T} \equiv \tfrac{1}{T} \textstyle\sum_{t \leq T} \left[\tfrac{N-a}{N} 1\{A_t = a\} + \tfrac{a+1}{N} 1\{A_t = a+1\}\right]$$

for $0 \leq a \leq N-1$. (See equation (A5) in Appendix A for details.) Finally, calculate the test statistic by:

$$\hat{\tau}_T \equiv \textstyle\sum_{a=0}^{N-1} \hat{\zeta}_{a,T} \hat{\rho}_{a,T} - \tfrac{1}{T} \sum_{t \leq T} \tilde{K}_t.$$

The next proposition establishes the asymptotic property of the test statistic $\hat{\tau}_T$.

Assumption 2 *(i) The elements in $(V_1, V_2, ..., V_N, K, \epsilon)$ are mutually independent with finite second moments and $E(\epsilon) = 0$. (ii) The marginal density for V_i is the same for all i and bounded above, and is bounded away from zero by a positive constant. The marginal distribution F_V satisfies $\int_r^{\overline{v}} [1 - F_V(t)]^{-1} dt < \infty$. (iii) $0 < \lambda_k^* < 1$ for all $k \in [\underline{k}, \overline{k}]$.*

Condition (i) strengthens Assumption 1. The independence between private values and entry costs is essential for the identification result in Corollary 1 while the orthogonality between ϵ and K is not indispensable for identification but helps to simplify the limiting distribution of our test statistic. Condition (ii) contains a mild restriction on the behavior of the distribution of private values F_V near the upper bound of its support. It holds, for example, if $F_V(t) \equiv 1 - \sqrt{1-t}$ for $t \in [\underline{v}, \overline{v}] \equiv [0, 1]$ and $0 < r < 1$. Such a tail condition ensures our test statistic is smooth in the conditional distribution of transaction prices $F_{W|A=m}$. Condition (iii) rules out extremely high or low entry costs that induce degenerate entry behaviors.

Proposition 3 *Suppose Assumptions 1 and 2 hold. Then $\sqrt{T}(\hat{\tau}_T - \tau_0)$ converges in distribution to a univariate normal distribution with a zero mean and a finite variance.*

For a given level α, let $\hat{c}_{1-\alpha/2,T}$ denote an estimator for the $100 \cdot (1 - \alpha/2)$-percentile of the limiting distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ using bootstrap procedures. (See Appendix C for the definition of $\hat{c}_{1-\alpha/2,T}$.) The decision rule for the test is to reject H_0 in favor of H_A (or H_L) if $\sqrt{T}\hat{\tau}_T > \hat{c}_{1-\alpha/2,T}$ (or if $\sqrt{T}\hat{\tau}_T < -\hat{c}_{1-\alpha/2,T}$); and do not reject H_0 otherwise.[6]

The next proposition shows the test is consistent against fixed alternatives, and attains the correct asymptotic level under the null. Let $\Pr(\sqrt{T}\hat{\tau}_T \leq \cdot \mid \tau_0 = \tau^*)$ denote the distribution of $\sqrt{T}\hat{\tau}_T$ conditioning on the true value for τ_0 being τ^*.

Proposition 4 *Under Assumptions 1 and 2,*

$$\lim_{T\to+\infty} \Pr\left(\sqrt{T}\hat{\tau}_T > \hat{c}_{1-\alpha/2,T} \mid \tau_0 = \tau^*\right) = 1 \ \forall \tau^* > 0; \tag{5}$$

$$\lim_{T\to+\infty} \Pr\left(\sqrt{T}\hat{\tau}_T < -\hat{c}_{1-\alpha/2,T} \mid \tau_0 = \tau^*\right) = 1 \ \forall \tau^* < 0; \tag{6}$$

$$\lim_{T\to+\infty} \Pr\left(\sqrt{T}\hat{\tau}_T > \hat{c}_{1-\alpha/2,T} \ \text{or} \ \sqrt{T}\hat{\tau}_T < -\hat{c}_{1-\alpha/2,T} \mid \tau_0 = 0\right) = \alpha. \tag{7}$$

Two facts are used for showing these results. First, the empirical distribution of $\sqrt{T}\left(\hat{\tau}_{T,b} - \hat{\tau}_T\right)$ provides a consistent estimator for the limiting distribution of $\sqrt{T}\left(\hat{\tau}_T - \tau_0\right)$ under the stated conditions, which is verified in Appendix C using results from Beran and Ducharme (1991). Second, $\sqrt{T}\tau_0$ is zero under the null but diverges to positive (or negative) infinity under the alternative.[7]

4.3 Mismeasurement of N

The number of potential bidders N could also be mismeasured in the data for several reasons, each of which has a different impact on the performance of our test.

First, under-measurement of N could happen due to finite-sample limitations. For example, consider the case where equilibrium entry probabilities are low and the set of potential bidders is measured as the union of entrants observed in the sample. In such a case, a potential bidder may not be counted in because it never entered as an active bidder in the finite sample. Our estimator for entry probabilities is thus biased upward under this mismeasurement, because potential bidders who decide not to enter in any auction reported in data are not counted in the denominator. On the other hand, this mismeasurement does not impact the estimation of the private value distribution, since the number of entrants is correctly reported in data.

[6] Our bootstrap inference uses an asymptotically non-pivotal statistic $\sqrt{T}\left(\hat{\tau}_T - \tau_0\right)$. One could construct asymptotically pivotal statistics using the pre-pivoting approach. This would help attain asymptotic refinements in the approximation of the distribution of the test statistic relative to a first-order asymptotic approximation or a bootstrap procedure using asymptotically non-pivotal statistics. This is computationally intensive due to bootstrap iterations and therefore we do not pursue this approach here.

[7] The web supplement (Fang and Tang (2014)) provides some Monte Carlo evidence for the finite-sample performance of our test under a slightly different specification where the value distribution is continuous over $[\underline{v}, \overline{v})$ with a probability mass at $\overline{v}$. It also includes some simulation results for the test performance under systematic mismeasurement of the entry cost.

To see how this affects our test, recall bidders' risk premia are weighted sums of interim profits from entry. Such an under-measurement in N has no impact on the estimation of these interim profits, which only depend on the private value distribution, the number of competing entrants A_{-i}, and the expectation of entry costs. On the other hand, it does affect estimates for the weights, or probability masses for A_{-i}, through its impact on estimates of entry probabilities. Unfortunately the sign of the resultant bias in the estimator for risk-premia is indeterminate, for it depends on the way those interim profits vary with A_{-i}, and how the distribution of A_{-i} changes with entry probabilities.[8] Nevertheless, as N increases in the data-generating process, the effect of under-measurement on the bias in the estimator for entry probabilities and risk premium decreases.

The second possibility for mismeasurement is over-counting N due to problematic assumptions. Again, consider the case where potential bidders are measured as the union of entrants observed in data. An implicit assumption is that a bidder who shows up in one auction is by default also a potential bidder in other auctions in data. This risks over-counting N when there are bidders who pay the costs to enter in some auctions but are not interested as a potential bidder in the others, say, due to budget or time constraints. Such over-counting leads to a *downward* bias in the estimator for entry probabilities but has no bearing on the estimation of interim profits. Again, the ultimate impact of such a bias on test performance is indeterminate, but diminishes as the real N increases.

Yet a third scenario is that the number of potential bidders reported in data, denoted by $\tilde{N}$, is a persistent random under-measurement of the real N in the DGP. In this case, the conditional distribution of $\tilde{N}$ given N, denoted by $\tilde{g}(\tilde{N} \mid N)$, is an additional parameter to be identified. This differs qualitatively from the first possibility (under-measurement due to finite sample limitations) in that the distribution $\tilde{g}$ is a model primitive that does not vary with the sample size. An, Hu and Shum (2010) studied the estimation of a related model of first-price auctions where the number of actual bidders are misclassified due to the truncation by binding reserve prices. It turns out we can apply their argument to the current context to identify the joint distribution of $(N, \tilde{N})$, and the distribution of transaction prices given N. Consequently, the entry probabilities conditional on N are also identified and tests for risk attitudes can be constructed as before.

For easy exposition of how the argument in An, Hu and Shum (2010) can be applied here, fix an entry cost and let $\tilde{N} = \tilde{f}(N, \tilde{\nu})$ and $N = f(\tilde{Z}, \nu)$, where $\tilde{Z}$ are some instruments that do not affect the distributions of bidders' private values. Assume bidders' private values, $\tilde{\nu}$ and $\tilde{Z}$ are mutually independent conditional on N. This implies that the transaction price

[8]Knowing the estimators for entry probabilities are biased upward under this mismeasurement is not sufficient for deriving the sign of bias in our estimator for risk-premium. This latter sign is determined by primitive elements and endogenous objects in the model (i.e. the utility function, the private value distribution, and the true equilibrium entry probabilities).

W is independent of $\tilde{N}$ and $\tilde{Z}$ given N, and that $\tilde{N}$ is independent of $\tilde{Z}$ given N. Provided the joint support of $(\tilde{Z}, \tilde{N})$ satisfy a mild full-rank condition, the conditional distribution of $\tilde{N}$ given N, the conditional distribution of W given N, and the joint distribution of $(\tilde{Z}, N)$ are jointly identified using a typical matrix diagonalization argument. (See Theorem 1 in An, Hu and Shum (2010) for a proof.) Knowledge of these distributions, together with the directly identifiable distribution of W given the number of entrants, implies that the entry probabilities are identified and consistently estimable. The proposed test for bidders' risk attitudes can then be constructed. We leave the implementation of a test based on such an idea for future research.

5 Inference of Risk Attitudes with Unobserved Entry Costs

In lots of applications, auctions reported in the data differ in observed characteristics of the auctioned object. Besides, the number of potential bidders N vary across auctions. In this section, we show how to use these sources of variation to infer bidders' risk attitudes, when entry costs known to bidders are *not* reported in data.

Assumption 3 *For all i, $V_i = h(Z) + \eta_i$ where Z is a vector of auction characteristics reported in the data, and η_i are i.i.d. draws from some distribution F_η independent of Z and N. The support of V_i is contained in $\mathbb{R}_+$.*

The function $h(\cdot)$ is a model primitive unknown to the econometrician. It is directly identifiable under a location normalization $E[\eta] = 0$. To see this, let $F_{V|Z}$ denote the distribution of V_i conditional on Z, where the subscript i is dropped due to symmetry. By the same argument used in Proposition 1 in Section 4.1, $F_{V|Z}$ is identified from the distribution of transaction prices $F_{W|Z,A=m}$. Assumption 3 and $E[\eta] = 0$ then imply $h(z)$ is identified as $h(z) = E[V_i \mid Z = z] = \int v dF_{V|Z=z}(v)$ for all z. The support condition that $\Pr(V_i \geq 0) = 1$ is not indispensable to our identification argument below. It can be replaced by an alternative condition that requires $u(0) \neq E[u((\eta_i - \eta_j)_+ - k)]$, which is a mild restriction on the model elements k, $u(.)$ and F_η.

Independence of η from N in Assumption 3 is analogous to the assumption of exogenous variation in the number of potential bidders in Haile, Hong and Shum (2004) and Guerre, Perrigne and Vuong (2009). As shown in Guerre, Perrigne and Vuong (2009), it is possible to identify bidders' risk attitudes even when N is endogenous, as long as the data contains valid instruments once conditioning on some control variables. We provide detailed discussions about this in Section 5.4.1.

Under Assumption 3, a test for risk attitudes can be constructed even when entry costs are not reported in the data. The idea is to exploit the fact that bidders' risk attitudes affect how entry probabilities vary with Z and N in equilibrium. To illustrate this idea, it is instructive to investigate the indifference condition in equilibrium. Such conditions equate bidders' utility from the certainty equivalent, $u(0)$, with their ex ante utility from entry. By construction ex ante utility from entry is a weighted average of "*interim*" utility, which conditions on the number of competing entrants with expectations taken with respect to private values. Entry probabilities enter ex ante utilities in the indifference condition through the weights assigned to interim utilities. These weights correspond to probability mass functions for the number of competing entrants. As Z and N vary, the entry probabilities (and therefore the weights) change endogenously in order to respect indifference condition in equilibrium.

If bidders are risk-neutral, then variation in Z induce the same rate of changes in interim utilities as that in expected private values. On the other hand, this equality fails when bidders are risk-averse (or risk-loving) due to the decreasing (or increasing) rate of increase in utility. Under the orthogonality and additive separability conditions in Assumption 3, the rate of changes in expected private values is over-identified as the ratio between changes in expected transaction prices. Thus, a test for bidders' risk attitudes would be feasible if the rate of changes in interim utilities can be related to the distribution of entry decisions and transaction prices.

We construct a linear system that characterizes the indifference conditions for various Z and N. Under Assumption 3, the number of unknowns in the system (i.e. the interim utilities) increases at the same pace as the number of equations. As long as the matrix of coefficients in the linear system (i.e. weights for interim utilities under various Z and N) has full-rank, we can recover the rates of changes in interim utilities from the distribution of prices and entry decisions.[9] This allows us to conduct a test for risk attitudes when the entry costs are not reported in the data.

5.1 Identification

Suppose the entry cost k is fixed across auctions but not reported in data.[10] To simplify exposition, assume there is no binding reserve price. Let $\lambda_{z,n}$ denote bidders' equilibrium entry probabilities in auctions with $Z = z$ and the number of potential bidders $N = n$. For $0 \leq a \leq n-1$, define:

$$\rho_{a,z,n} \equiv \Pr(A_{-i} = a \mid Z = z, N = n) = \binom{n-1}{a} (\lambda_{z,n})^a (1 - \lambda_{z,n})^{n-a-1}. \tag{8}$$

[9] See Assumption 3 and discussions after Lemma 2 below for details about the sufficent rank condition.

[10] We could generalize this section by allowing entry costs to vary across auctions in the data-generating process as well. To do this, we could let the vector of auction heterogeneity consist of two subvectors $(Z, \hat{Z})$, and let k be an *unknown* function of $\hat{Z}$ alone. Our inference method below applies as long as we could condition on $\hat{Z}$ and exploit the variations in Z.

In the mixed-strategy equilibrium in the entry stage,

$$u(0) = E\left[u((V_i - P_i)_+ - k) \mid Z = z, N = n\right] = \sum_{a=0}^{n-1} \psi_a(z)\rho_{a,z,n} \tag{9}$$

where $\psi_a(z) \equiv E[u((V_i - P_i)_+ - k) \mid A_{-i} = a, Z = z]$, and A_{-i}, V_i, P_i are defined as in Section 3. Under Assumptions 3, $(V_i - P_i)_+$ does not depend on n given z and the number of competing entrants $A_{-i} = a$. For all $a \geq 1$, $\psi_a(z)$ is defined as:

$$\psi_a(z) \equiv E[u((V_i - P_i)_+ - k) \mid Z = z, A_{-i} = a] = E\left[u\left(\left(\eta_i - \eta^{(a:a)}\right)_+ - k\right)\right],$$

where $\eta^{(a:a)}$ is the largest among a independent draws from F_η. Also, by construction,

$$\psi_0(z) = E[u(V_i - k) \mid Z = z] = E[u(h(z) + \eta_i - k)]$$

where the second equality follows from independence between η_i and z. Clearly $\psi_a(z)$ does not depend on z for $a \geq 1$; thus we let $\psi_a^* \equiv \psi_a(z)$ for all $a \geq 1$ and any z.

Assumption 4 *There exist z, z', z'' on the support of Z and some n such that $0 < \lambda_{z,s} < 1$ for all $2 \leq s \leq n$, and $\lambda_{z',n}, \lambda_{z'',n} \in (0,1)$.*

Assumption 4 requires there be enough variation in auction characteristics and in potential competition in data. The condition that entry probabilities are in $(0,1)$ rules out uninteresting cases with degenerate entry behaviors.[11] This condition can be directly verified from the data in principle.

The characterization of entry probabilities for $Z = z$ and $N = s$ for $2 \leq s \leq n$ leads to a system of $(n-1)$ equations:

$$\psi_0(z)\rho_{0,z,s} + \sum_{a=1}^{s-1} \psi_a^* \rho_{a,z,s} = u(0) \text{ for } s = 2, .., n. \tag{10}$$

We also have additional equations from auctions with different observed features z' or z'' and $N = n$. That is,

$$\psi_0(z')\rho_{0,z',n} + \sum_{a=1}^{n-1} \psi_a^* \rho_{a,z',n} = u(0); \text{ and likewise with } z' \text{ replaced by } z''. \tag{11}$$

Stacking the system of equations from (10)-(11) and moving the terms that involve ψ_1^* to the right, we have:

$$\underbrace{\begin{pmatrix} \rho_{0,z'',n} & 0 & 0 & \rho_{2,z'',n} & \cdots & \rho_{n-1,z'',n} \\ 0 & \rho_{0,z',n} & 0 & \rho_{2,z',n} & \cdots & \rho_{n-1,z',n} \\ 0 & 0 & \rho_{0,z,2} & 0 & \cdots & 0 \\ 0 & 0 & \rho_{0,z,3} & \rho_{2,z,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \rho_{0,z,n} & \rho_{2,z,n} & \cdots & \rho_{n-1,z,n} \end{pmatrix}}_{(n+1)\text{-by-}(n+1)} \underbrace{\begin{pmatrix} \psi_0(z'') \\ \psi_0(z') \\ \psi_0(z) \\ \psi_2^* \\ \vdots \\ \psi_{n-1}^* \end{pmatrix}}_{(n+1)\text{-by-}1} = \underbrace{\begin{pmatrix} u(0) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u(0) \end{pmatrix}}_{(n+1)\text{-by-}1} - \underbrace{\begin{pmatrix} \rho_{1,z'',n} \\ \rho_{1,z',n} \\ \rho_{1,z,2} \\ \vdots \\ \vdots \\ \rho_{1,z,n} \end{pmatrix}}_{(n+1)\text{-by-}1} \psi_1^*. \tag{12}$$

[11] This assumption restricts how the certainty equivalent $u(0)$ is compared to bidders' interim utility given z and the number of competing entrants. For instance, in the case with $N = 2$, this condition suggests the certainty equivalent $u(0)$ is bounded between $\psi_0(z)$ and $\psi_1(z)$ at least for some z.

The linear system has $n+2$ unknowns (i.e. $\psi_0(z'')$, $\psi_0(z')$, $\psi_0(z)$ and $\{\psi_a^* : 1 \le a \le n-1\}$) and $n+1$ equations. Nonetheless, the next lemma shows how (12) can be used to relate the ratio of changes in interim utilities ψ_0 to entry probabilities as z varies.

Under Assumption 4, the coefficient matrix in (12) has a full rank of $n+1$. (See the proof of Lemma 2 in Appendix A.) Now replace $u(0)$ by 0 and replace ψ_1^* by some arbitrary nonzero constant c, and solve for the $n+1$ unknowns $(\psi_0(z''), \psi_0(z'), \psi_0(z), \{\psi_a^* : 2 \le a \le n-1\})$ in (12). Denote the unique solutions by $(\tilde{\psi}_{0,z''}, \tilde{\psi}_{0,z'}, \tilde{\psi}_{0,z}, \{\tilde{\psi}_a : 2 \le a \le n-1\})$.

Lemma 2 *Suppose Assumptions 3-4 hold. If $h(z') \neq h(z'')$, then*

$$\frac{\psi_0(z') - \psi_0(z)}{\psi_0(z'') - \psi_0(z')} = \frac{\tilde{\psi}_{0,z'} - \tilde{\psi}_{0,z}}{\tilde{\psi}_{0,z''} - \tilde{\psi}_{0,z'}}. \tag{13}$$

Assumption 4 is sufficient but not necessary for (13) to hold under Assumption 3. We could relax Assumption 4 and only require the data-generating process to contain enough variation in Z and N so that a linear system similar to (12) can be constructed for at least three different values of Z with the coefficient matrix in (12) being non-singular.[12] The next proposition states a testable implication of bidders' risk attitudes when entry costs are not recorded in data.

Proposition 5 *(a) Under Assumption 3,*

$$h(z_1) - h(z_2) = E(W \mid Z = z_1, A = a) - E(W \mid Z = z_2, A = a)$$

for any (z_1, z_2) and $a \ge 2$. (b) Suppose Assumptions 3-4 hold for (z, z', z'') such that $h(z'') > h(z') > h(z)$. Then

$$\frac{\tilde{\psi}_{0,z'} - \tilde{\psi}_{0,z}}{\tilde{\psi}_{0,z''} - \tilde{\psi}_{0,z'}} \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \frac{h(z') - h(z)}{h(z'') - h(z')} \text{ iff } u'' \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} 0. \tag{14}$$

To construct a test based on Proposition 5, one needs to locate a triple (z, z', z'') such that $h(z'') > h(z') > h(z)$. Under Assumption 3, this sequence of strict inequalities is equivalent to $E[W \mid z'', A = a] > E[W \mid z', A = a] > E[W \mid z, A = a]$ for all $a \ge 2$. (See part (a) of Proposition 5.) Thus such a triple can be found using the distribution of prices conditional

[12] Specifically, a weaker sufficient condition for Lemma 2 is as follows: *"There exist $J \ge 3$ values of Z, denoted $\{z^j : 1 \le j \le J\}$, and J overlapping sets of integers on the support of N, each of which is denoted $\zeta^j \equiv \{n^{j,1}, n^{j,2}, .., n^{j,M_j}\}$ with $M_j \equiv \#\{\zeta^j\}$, such that the matrix of coefficients for $\{\psi_0(z^j) : j \le J\}$ and $(\psi_2^*, ., \psi_{\bar{n}-1}^*)$ (where $\bar{n} \equiv \max\{\cup_{j \le J}\{\zeta^j\}\}$) has full-rank in the linear system of $\sum_{j \le J} M_j$ equations characterizing the equilibrium entry probabilities for various Z and N."* This condition necessarily requires $\sum_{j \le J} M_j \ge \bar{n} + J - 2$. Note Assumption 4 is a special case of this condition with $J = 3$, $(z^1, z^2, z^3) = (z, z', z'')$, $\zeta^1 = \{2, 3, .., n\}$, and $\zeta^2 = \zeta^3 = \{n\}$.

on Z and the number of entrants. Note this does not require the location normalization $E[\eta_i] = 0$.

In some other contexts, it is possible to locate such a triple using the shape restrictions on h that are known *a priori* to researchers. For example, economic theory or common sense sometimes restricts $h(.)$ to be monotonic in one of the elements in Z, or suggests $F_{V|Z}$ is stochastically ordered over a known triple $Z \in \{z, z', z''\}$. In such cases, the choice of the triple to be used in the test is immediate.[13] In the following subsection, we assume a triple z, z' and z'' with $h(z'') > h(z') > h(z)$ is known and fixed.

5.2 Test Statistic

We now construct a test statistic based on Proposition 5 and some triple (z, z', z'') and n known to satisfy Assumption 4. First, estimate the coefficient matrix on the right-hand side in (12) and the coefficient vector on the left-hand side in (12) by plugging in sample analogs of the entry probabilities in (8). Next, replace $u(0)$ with 0 and ψ_1^* with an arbitrary negative constant c in (12) and solve it to get estimates for $(\tilde{\psi}_{0,z''}, \tilde{\psi}_{0,z'}, \tilde{\psi}_{0,z}, \{\tilde{\psi}_a : 2 \leq a \leq n-1\})$.[14] Denote these estimates by $\hat{\psi}_0(z)$, $\hat{\psi}_0(z')$, $\hat{\psi}_0(z'')$ and $\{\hat{\psi}_a^* : 2 \leq a \leq n-1\}$ respectively. Our test statistic is

$$\tilde{\tau} \equiv \frac{\hat{\psi}_0(z') - \hat{\psi}_0(z)}{\hat{\psi}_0(z'') - \hat{\psi}_0(z')} - \hat{R}$$

where

$$\hat{R} \equiv \sum_{a=2}^{n-1} \hat{q}_a \left(\frac{\hat{E}(W \mid Z = z', A = a) - \hat{E}(W \mid Z = z, A = a)}{\hat{E}(W \mid Z = z'', A = a) - \hat{E}(W \mid Z = z', A = a)} \right) \tag{15}$$

with $\hat{E}(W \mid Z = z, A = a)$ being kernel estimators for expected transaction prices given z and the number of entrants $A = a$; and $\{\hat{q}_a : 2 \leq a \leq n-1\}$ are data-dependent weights. In the simulations below, we use $\hat{q}_a \equiv \hat{\sigma}_a^{-2}/(\sum_{s=2}^{n-1} \hat{\sigma}_s^{-2})$, where $\hat{\sigma}_a$ is the standard error for the ratio of differences in the parenthesis on the right-hand side of (15).

We now briefly discuss the asymptotic property of our test statistic $\tilde{\tau}$. It is instructive to look at the case where the support of Z is discrete, which is also what we consider in the simulations of Section 5.3 below. In this case, $\tilde{\tau}$ is by construction a smooth function of sample averages. (To see this, note $\hat{\lambda}_{z,n}$ are sample averages and $\hat{E}(W \mid Z = z, A = a)$ are ratios whose numerators and denominators are sample averages. It then follows that $\hat{\psi}_0(z)$, $\hat{\psi}_0(z')$, $\hat{\psi}_0(z'')$ and $\hat{R}$ are smooth functions of sample averages.) Under Assumptions 3 and

[13] In other cases, we would need a preliminary step for selecting a triple satisfying the inequalities, based on comparing the estimates of $E(W \mid Z = z,\ A = a)$. We leave issues such as how to account for the impact of pretesting in inference to future research.

[14] To identify the ratio of differences between interim utilities ψ_0 under various z, the sign of the chosen constant c does not matter. However, using a negative c has the additional benefit of recovering the correct ordering of ψ_0 under different z. (See the proof of Lemma 2 in Appendix A for details.)

4 and the conditions on finite second moments of transaction prices, the Delta Method can be applied to show that $\tilde{\tau}$ converges at the parametric rate to its population counterpart τ_*, which is defined as the difference between the two ratios compared in (14). Furthermore, the asymptotic distribution of $\tilde{\tau}$ is normal with a zero mean.

A test for risk attitudes follows a procedure similar to that of Section 4.2, with critical values estimated using bootstrap resampling. With $\tilde{\tau}$ being a smooth function of sample averages, the bootstrap estimator is expected to be consistent (see Section 2.1 in Horowitz (2001) for the definition of bootstrap consistency). For the same reason, the test using bootstrap critical values is expected to perform at least as well as one based on first-order asymptotic approximation both in terms of the errors in rejection probabilities and its asymptotic power.

Finally, note that if several distinct triples on the support of Z are known to satisfy Assumption 4, then we can construct a more efficient version of the test. For example, the test statistic could be replaced by some form of an average of several $\tilde{\tau}$'s, each of which is calculated based on one of the triples.[15]

5.3 Monte Carlo Simulation

This subsection presents some Monte Carlo evidence for the performance of the test above in finite samples. The data-generating process (DGP) is as follows. The distribution of auction characteristics Z is multinomial over a discrete support $\{1,2,3\}$ with equal probability masses. Upon entry, a bidder i's private value is $\beta_1 + \beta_2 Z + \eta_i$, where $\beta \equiv (\beta_1, \beta_2)$ are parameters to be experimented with and η_i are i.i.d draws from a uniform distribution over $[-2,2]$. The support of potential bidders is $\{2,3,4,5,6\}$. Conditional on Z, the distribution of N is stochastically increasing: $\Pr(N = n|Z = 1) = \frac{1}{3}$ for $n = 2$ and $\frac{1}{6}$ for any $n \neq 2$; $\Pr(N = n|Z = 2) = \frac{1}{3}$ for $n = 4$ and $\frac{1}{6}$ for any $n \neq 4$; and $\Pr(N = n|Z = 3) = \frac{1}{3}$ for $n = 6$ and $\frac{1}{6}$ for any $n \neq 6$. Such a specification is meant to capture the possibility that the value of the auctioned object is positively correlated with potential competition. An alternative DGP where the distribution of N is restricted to be invariant in Z produces similar results. Bidders' von-Neumann-Morgenstern utility is $u(c) \equiv \left(\frac{1+c}{10}\right)^{\gamma}$. The entry costs observed by potential bidders is fixed at $K = 1$ in all auctions. As explained in Section 5.1, the test does not require knowledge of the actual entry cost in the data.

Table 1 reports equilibrium entry probabilities for $\beta = (2,5)$ and $(2,3)$ respectively. As the theoretical model suggests, for a fixed β, the entry probabilities are monotonically decreasing in N and increasing in Z (due to the monotonicity of $h(Z)$). Also, as the marginal effect of Z diminishes from $\beta_2 = 5$ to $\beta_2 = 3$, bidders are less likely to enter the bidding stage for any given Z and N. Under the current utility specification, risk aversion appears to result in lower entry probabilities *ceteris paribus*. Most importantly, the entry probabilities

[15] We thank the Associate Editor for pointing this out.

are in the interior of $(0,1)$. This implies the non-singularity of the coefficient matrix in (12).

Table 1(a): Equilibrium Entry Probabilities: $\beta = (2,5)$

	$\gamma = 0.6$			$\gamma = 1$			$\gamma = 1.4$		
	$Z = 1$	$Z = 2$	$Z = 3$	$Z = 1$	$Z = 2$	$Z = 3$	$Z = 1$	$Z = 2$	$Z = 3$
$N = 2$	0.7767	0.8440	0.8756	0.9000	0.9429	0.9600	0.9547	0.9789	0.9871
$N = 3$	0.5120	0.5881	0.6299	0.6667	0.7449	0.7851	0.7714	0.8420	0.8756
$N = 4$	0.3763	0.4419	0.4797	0.5145	0.5930	0.6365	0.6213	0.7034	0.7468
$N = 5$	0.2966	0.3524	0.3854	0.4164	0.4883	0.5297	0.5152	0.5961	0.6412
$N = 6$	0.2445	0.2927	0.3216	0.3491	0.4139	0.4520	0.4386	0.5147	0.5585

Table 1(b): Equilibrium Entry Probabilities: $\beta = (2,3)$

	$\gamma = 0.6$			$\gamma = 1$			$\gamma = 1.4$		
	$Z = 1$	$Z = 2$	$Z = 3$	$Z = 1$	$Z = 2$	$Z = 3$	$Z = 1$	$Z = 2$	$Z = 3$
$N = 2$	0.7185	0.7960	0.8349	0.8571	0.9130	0.9375	0.9267	0.9626	0.9762
$N = 3$	0.4556	0.5323	0.5768	0.6051	0.6882	0.7337	0.7119	0.7914	0.8323
$N = 4$	0.3299	0.3934	0.4319	0.4573	0.5354	0.5813	0.5590	0.6435	0.6914
$N = 5$	0.2580	0.3110	0.3438	0.3660	0.4352	0.4774	0.4568	0.5366	0.5840
$N = 6$	0.2118	0.2569	0.2852	0.3045	0.3658	0.4039	0.3852	0.4585	0.5031

We report performance of the test in simulated samples that contain $T = 3,000$ or $T = 5,000$ auctions with variation in Z and N. For each pair (β, γ) and a sample size T, we simulate $S = 400$ samples. For each simulated sample, we calculate a test statistic $\tilde{\tau}$ and record a decision under significance levels $\alpha \in \{5\%, 10\%, 15\%\}$ respectively, based on critical values estimated from $B = 400$ bootstrap samples. Table 2 summarizes the performance of the test for $\gamma \in \{0.6, 0.7, 0.8, 1.0, 1.2, 1.3, 1.4\}$.

Each row in Table 2 corresponds to a DGP with a pair (β, γ) and a sample size T. The numbers in each cell are the proportions of S simulated samples in which H_L, H_0, H_A are accepted respectively.

Table 2 (a) reports the test results when the auction heterogeneity Z has a larger marginal impact on bidders' values ($\beta_2 = 5$). With a moderate sample size $T = 3,000$, the probabilities for rejecting the null is reasonably close to the targeted significance levels α under the null. Errors in rejection probabilities also decrease as the sample size increases to $T = 5,000$. For both sample sizes and all significance levels , the power of the test are reasonably high for almost all alternatives except $\gamma = 0.8$. Besides, the power is also shown to approach 1 as sample sizes increase. The probability for "Type-III" error (i.e. rejecting the null in favor of a wrong alternative) is practically zero across all specifications and sample sizes.

Table 2(a): Probabilities for Accepting $[H_L, H_0, H_A]$: $\beta = (2, 5)$

	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 15\%$
$T = 5,000$			
$\gamma = 0.6$	[0.00%, 0.75%, 99.25%]	[0.00%, 0.75%, 99.25%]	[0.00%, 0.75%, 99.25%]
$\gamma = 0.7$	[0.00%, 16.00%, 84.00%]	[0.00%, 8.00%, 92.00%]	[0.00%, 4.25%, 95.75%]
$\gamma = 0.8$	[0.00%, 80.75%, 19.25%]	[0.00%, 53.75%, 46.25%]	[0.00%, 37.00%, 63.00%]
$\gamma = 1$	[4.50%, 94.00%, 1.50%]	[6.25%, 89.25%, 4.50%]	[10.75%, 83.50%, 5.75%]
$\gamma = 1.2$	[56.25%, 43.75%, 0.00%]	[62.25%,37.75%,0.00%]	[64.00%, 36.00%, 0.00%]
$\gamma = 1.3$	[85.75%, 14.25%, 0.00%]	[89.25%,10.75%,0.00%]	[91.75%, 8.25%, 0.00%]
$\gamma = 1.4$	[97.25%, 2.75%, 0.00%]	[97.75%, 2.25%, 0.00%]	[98.25%, 1.75%, 0.00%]
$T = 3,000$			
$\gamma = 0.6$	[0.00%, 6.75%, 93.25%]	[0.00%, 4.25%, 95.75%]	[0.00%, 3.00%, 97.00%]
$\gamma = 0.7$	[0.00%,38.00%,62.00%]	[0.00%,19.00%,81.00%]	[0.00%,13.50%,86.50%]
$\gamma = 0.8$	[0.00%,88.50%,11.50%]	[0.00%,77.75%,22.25%]	[0.00%,58.50%,41.50%]
$\gamma = 1.0$	[5.00%, 92.75%, 2.25%]	[7.25%, 88.50%, 4.25%]	[11.50%,82.25%,6.25%]
$\gamma = 1.2$	[48.50%,51.50%,0.00%]	[54.50%,45.50%,0.00%]	[58.00%,42.00%,0.00%]
$\gamma = 1.3$	[76.75%,23.25%,0.00%]	[80.00%,20.00%,0.00%]	[83.00%,17.00%,0.00%]
$\gamma = 1.4$	[95.25%, 4.75%, 0.00%]	[97.00%, 3.00%, 0.00%]	[98.25%, 1.75%, 0.00%]

Our empirical estimates for the power of the test are low against the risk-averse alternative $\gamma = 0.8$. For example, when the significance level is $\alpha = 5\%$ in Table 2(a), our empirical estimates for the power is 11.5% with $T = 3,000$ and 19.25% with $T = 5,000$. However, the power improves at a fast pace as the alternative moves farther away from the null. For example, in the same Table 2(a), when $\gamma = 0.7$ and $\alpha = 5\%$, the estimates for power under $T = 3,000$ and $5,000$ increase substantially to 62% and 84% respectively. Also, for γ's with the same distance from the null value 1, the power reported against risk-loving alternatives are considerably higher than those against risk-averse alteratives. For instance, in Table 2(a), when $\alpha = 5\%$ and $\gamma = 1.2$, the power is 48.5% for $T = 3,000$ and 56.25% for $T = 5,000$, which are a lot greater than their counterparts under $\gamma = 0.8$.

That the reported power can be low against the alternatives with $\gamma \in [0.8, 1)$ is due to the combination of two factors. First, these risk-averse alternatives correspond to utility functions with curvatures not too far from that of a linear function. Thus the difference between the ratios compared in Proposition 5 (i.e. τ_*, or the probability limit of $\tilde{\tau}$) is close to zero under these DGPs. On the other hand, the standard error for estimating conditional entry probabilities are large relative to the absolute value of this difference τ_* because of smaller sample sizes after Z and N are controlled for. (Recall the T observations consists of auctions with 15 possible combinations of (Z, N) in the DGP considered.) This in turn translates into relatively large standard errors for $\tilde{\tau}$. To sum up, the test statistics are asymptotically normal both under $\gamma = 1$ and $\gamma = 0.8$, with variances large relative to the difference between the means. This could explain the observed lower power against alternatives $\gamma \in [0.8, 1)$. We argue this should not be interpreted as evidence of unsatisfactory finite sample performance of our test. Rather it is due to the fact that the curvature of utility functions are close to being linear for $\gamma \in [0.8, 1)$. To reiterate, the power of the test does improve substantially either as the sample size increases, or as γ moves farther away from 1.

Table 2(b): Probabilities for Accepting $[H_L, H_0, H_A]$: $\beta = (2, 3)$

	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 15\%$
$T = 5,000$			
$\gamma = 0.6$	[0.00%,10.00%,90.00%]	[0.00%, 6.00%, 94.00%]	[0.00%, 4.25%, 95.75%]
$\gamma = 0.7$	[0.00%,47.25%,52.75%]	[0.00%,30.25%,69.75%]	[0.00%,23.00%,77.00%]
$\gamma = 0.8$	[0.00%,88.25%,11.75%]	[0.00%,79.00%,21.00%]	[0.00%,66.75%,33.25%]
$\gamma = 1.0$	[3.00%, 94.25%, 2.75%]	[5.75%, 89.50%, 4.75%]	[9.75%, 84.00%, 6.25%]
$\gamma = 1.2$	[39.25%,60.75%,0.00%]	[45.25%,54.75%,0.00%]	[50.50%,49.50%,0.00%]
$\gamma = 1.3$	[68.25%,31.75%,0.00%]	[72.75%,27.25%,0.00%]	[77.00%,23.00%,0.00%]
$\gamma = 1.4$	[87.25%,12.75%,0.00%]	[90.50%, 9.50%, 0.00%]	[93.00%, 7.00%, 0.00%]
$T = 3,000$			
$\gamma = 0.6$	[0.00%,15.25%,84.75%]	[0.00%, 8.75%, 91.25%]	[0.00%, 6.25%, 93.75%]
$\gamma = 0.7$	[0.00%,61.25%,38.75%]	[0.00%,41.25%,58.75%]	[0.00%,31.00%,69.00%]
$\gamma = 0.8$	[0.00%, 91.25%, 8.75%]	[0.00%,84.75%,15.25%]	[0.00%,77.75%,22.25%]
$\gamma = 1.0$	[4.75%, 93.00%, 2.25%]	[8.00%, 87.50%, 4.50%]	[10.75%,83.00%,6.25%]
$\gamma = 1.2$	[33.00%,67.00%,0.00%]	[41.00%,59.00%,0.00%]	[45.25%,54.75%,0.00%]
$\gamma = 1.3$	[62.75%,37.25%,0.00%]	[67.50%,32.50%,0.00%]	[73.50%,26.50%,0.00%]
$\gamma = 1.4$	[75.75%,24.25%,0.00%]	[80.75%,19.25%,0.00%]	[84.50%,15.50%,0.00%]

Table 2(b) reports the test performance when $\beta = (2, 3)$. Overall, it registers the same patterns as those in Table 2(a). More interestingly, a comparison between Tables 2(a) and 2(b) suggests that, for any T, α and a fixed alternative $\gamma \neq 1$, the power of the test is larger in the DGP with a greater marginal effect of auction heterogeneity. Such a pattern across the two panels is consistent with the idea that underlies our test: The identification of risk attitudes is driven by the difference between the rate of changes in interim utilities and the rate of changes in expected private values as Z varies exogenously. The distance between these two ratios under any alternative depends on the difference in expected private values as Z varies, which depends on how Z enters $h(Z)$.

5.4 Further Discussions

5.4.1 Endogeneity in the number of potential bidders

It is possible to extend the results from Sections 5.1-5.2 to the case where the number of potential bidders N are endogenous. The main idea is to use additional auction heterogeneity reported in data as instrumental variables under exclusion restrictions. Guerre, Perrigne and Vuong (2009) introduced an instrument-based argument to recover bidders' utility functions in first-price auctions. Below we discuss how their argument can be applied in the current context.

Let the number of potential bidders be given by a structural equation $N = \iota(Z, X, \epsilon)$, where (Z, X) are auction heterogeneity reported in data; and ϵ is some auction-level variable not reported in data. As before, we maintain $V_i = h(Z) + \eta_i$ for all i. Now assume the idiosyncratic component η_i is independent of (X, Z) conditional on ϵ. Conditional on ϵ, η_i's are independent draws from the same distribution $F_{\eta_i|\epsilon}$ with $E[\eta_i \mid \epsilon] = 0$.

Under these conditions, the number of potential bidders may remain endogenous even after conditioning on (Z, X), because of the dependence between ϵ and η_i even after conditioning on (Z, X). On the other hand, these conditions imply that η_i are orthogonal to both (Z, N) conditional on ϵ. Note X differs from Z in that X only affects potential competition but not bidders' private values. The existing literature abound in examples of such variables. In e-bay coin auctions studied in Bajari and Hortacsu (2005), sellers' reputation affects bidders' participation decisions but not their private values. In U.S. Forest Service auctions considered in Athey, Levin, and Seira (2004), road costs and the density of timber were found to affect participation but not bidders' values.[16]

Following Guerre, Perrigne and Vuong (2009), assume $\epsilon = N - E[N \mid Z, X]$. That is, ϵ can be recovered from data as the residual of the nonparametric regression of N on (Z, X). Our identification method from Section 5.1 applies once conditioning on ϵ. Specifically, the interim utilities $\psi_a(z, \epsilon)$ and the entry probabilities $\rho_{a,z,n}(\epsilon)$ on the right-hand side of (9) now all depend on ϵ, whose realized values can be recovered through regressions, and therefore can be controlled for. The identification of risk attitudes then follows from the same argument as in Proposition 5 after conditioning on ϵ.

5.4.2 Full identification of the utility function

We conclude this section with brief discussions about the possibility to use auction heterogeneity observed from data to fully recover bidders' utility functions. Consider a cross-sectional data with a large number of independent auctions, which share the same entry

[16] To be precise, the exclusion restriction in these papers are stated for entry decisions as opposed to decisions to become a *potential* bidder. Nevertheless, the arguments used in those papers can be applied here to justify the inclusion of these variables in X above.

costs k (unknown to econometricians) and the same number of potential bidders N, but differ in their auction-level heterogeneity Z. Then in the mixed-strategy Nash equilibrium in the entry stage,

$$\int u(y-k)dF_{Y|Z,N}(y \mid z) = u(0) \tag{16}$$

where $Y_i \equiv (V_i - P_i)_+$ and $F_{Y|Z,N}$ denotes its distribution given (Z, N). The distribution of Y depends on Z both directly through $F_{V_i|Z}$ and indirectly through equilibrium entry probabilities that affect the distribution of P_i through the distribution of the number of entrants A. Note for a given k, (16) is an integral equation in $u(.)$, with the kernel being the conditional density $f_{Y|Z,N}$ which is identified by the argument used in Proposition 1 in Section 4.1. Besides, with exogenous variation in Z and N, we can augment a system of integral equations of $u(.)$ by including additional equations similar to (16) that are derived for different N and Z. We conjecture it is possible to establish the uniqueness of the solutions of $u(.)$ and k in such a system of integral equations, after imposing location and scale normalizations (such as $u(0) = 0$ and $u'(0) = 1$) and additional primitive conditions that restricts how $F_{Y|Z,N}$ varies with (Z, N). We leave this as a direction for future research.[17]

6 Extensions

6.1 Affiliated Private Values

The assumption of independent private values is instrumental for recovering the distribution of private values from the distribution of transaction prices, because it implies a one-to-one mapping exists between the parent distribution and the distribution of the second-largest order statistic it generates. However, independence between private values fails in certain situations, e.g. if bidders' values are correlated through auction heterogeneity known to bidders but not reported in the data. With affiliated private values, the joint distribution of private values is no longer point identified from the distribution of transaction prices for a given N (see Athey and Haile (2002)). Nevertheless, Aradillas-Lopez, Gandhi and Quint (2013) construct sharp bounds on bidders' ex ante expected surplus, using exogenous variation in the number of active bidders (see Theorem 1 in Aradillas-Lopez, Gandhi and Quint (2013)).

In our model, exogenous variation in the number of entrants A follows from a more primitive condition of exogenous variation in the number of potential bidders N, and from the fact that mixed-strategy entry equilibria are non-selective. As a result, we can apply the partial identification result from Aradillas-Lopez, Gandhi and Quint (2013) to construct

[17] For any fixed k, let $\Gamma(u)(z) \equiv \int u(t-k)dF_{Y|Z,N}(t \mid z)$. That is, Γ is a linear operator mapping from the space of continuous, bounded functions over the support of Y into the space of continuous, bounded functions over the support of Z.

bounds on bidders' ex ante surplus from entry. Comparing these bounds with entry costs reported in data may reveal some information about the bidders' risk preference.

To see this, consider a model with no binding reserve price. Fix the number of potential bidders at n and the entry cost at k. If the joint distribution $F_{(V_1,...,V_n)|k}$ is affiliated and exchangeable in bidders' indices, then there exist symmetric mixed-strategy Nash equilibria in entry stage where bidders enter independently with probability λ_k^* as characterized in Lemma 1 (see Lemma A1 for details). In addition, we maintain the following assumption about exogenous variation in the number of *potential* bidders. Let $\bar{n}$ denote the largest integer on the support of the number of potential bidders N in the data-generating process.

Assumption 5 *For any k, the random vector $(V_i)_{i=1}^{\bar{n}}$ is affiliated and its joint distribution is continuous and exchangeable in bidders' identities over the support $[\underline{v},\bar{v}]^{\bar{n}}$. For any subset $\tilde{n} \subseteq \{1,2,...,\bar{n}\}$, the joint distribution $F_{(V_i)_{i\in\tilde{n}}|k}$ equals the corresponding marginal distribution derived from $F_{(V_i)_{i=1}^{\bar{n}}|k}$.*

Recall that a mixed-strategy entry equilibrium is non-selective in that, for any set of potential bidders $\tilde{n}$ and any set of entrants $\tilde{a} \subseteq \tilde{n}$, the joint distribution of private values $F_{(V_i)_{i\in\tilde{a}}|k}$ equals the corresponding marginal distribution derived from $F_{(V_i)_{i\in\tilde{n}}|k}$. Hence this assumption implies that the variation in the number of entrants is exogenous in the sense that it is not correlated with the distribution of private values. Thus the approach in Aradillas-Lopez, Gandhi and Quint (2013) can be applied to bound the *ex ante* expected profits from entry. Let $\zeta_a(k)$ denote the expected surplus $(V_i - P_i)_+$ for i conditioning on competing with $A_{-i} = a$ other entrants and an entry cost k.

Proposition 6 *(An Application of Theorem 1 in Aradillas-Lopez, Gandhi and Quint (2013).) Under Assumption 5, $\zeta_a^L(k) \leq \zeta_a(k) \leq \zeta_a^U(k)$ for any k and $0 \leq a < \bar{n}-1$, where*

$$\begin{aligned}
\zeta_a^L(k) &\equiv \frac{1}{a+1}\left(\int_{\underline{v}}^{\bar{v}} t dF^+_{V^{(a+1:a+1)}|k,a+1}(t) - \int_{\underline{v}}^{\bar{v}} t dF_{W|k,A=a+1}(t)\right) \text{ and} \\
\zeta_a^U(k) &\equiv \frac{1}{a+1}\left(\int_{\underline{v}}^{\bar{v}} t dF^-_{V^{(a+1:a+1)}|k,a+1} - \int_{\underline{v}}^{\bar{v}} t dF_{W|k,A=a+1}(t)\right),
\end{aligned}$$

where $F_{W|k,A=s}$ is the distribution of transaction price given k and $A=s$, and

$$\begin{aligned}
F^+_{V^{(s:s)}|k,s}(t) &= \textstyle\sum_{m=s+1}^{\bar{n}} \frac{s}{m(m-1)} F_{V^{(m-1:m)}|k}(t) + \frac{s}{\bar{n}} F_{V^{(\bar{n}-1:\bar{n})}|k}(t) \\
F^-_{V^{(s:s)}|k,s}(t) &= \textstyle\sum_{m=s+1}^{\bar{n}} \frac{s}{m(m-1)} F_{V^{(m-1:m)}|k}(t) + \frac{s}{\bar{n}} \left[\phi_{\bar{n}}^{-1}(F_{V^{(\bar{n}-1:\bar{n})}|k}(t))\right]^{\bar{n}}.
\end{aligned}$$

The intuition of this result is as follows. First, an application using the Law of Total Probability reveals:

$$F_{V^{(a+1:a+1)}|k} = \tfrac{1}{a+2} F_{V^{(a+1:a+2)}|k} + \tfrac{a+1}{a+2} F_{V^{(a+2:a+2)}|k}.$$

Then recursive substitutions show that $F_{V^{(a+1:a+1)}|k}$ (which is not point-identified from the distribution of transaction prices due to affiliation) can be written as a linear combination of the distribution of second-largest order statistic $F_{V^{(m-1:m)}|k}$ for $m = a+2, .., \bar{n}$ (which is directly identifiable from the data) and the distribution of the largest order statistic $F_{V^{(\bar{n}:\bar{n})}|k}$. Second, it can be shown that the distribution of $V^{(\bar{n}:\bar{n})}$ given k is bounded between $[\phi_{\bar{n}}^{-1}(F_{V^{(\bar{n}-1:\bar{n})}|k}(.))]^{\bar{n}}$ and $F_{V^{(\bar{n}-1:\bar{n})}|k}(.)$. These bounds are derived by exploiting the link between the distribution of the first-order and the second-order statistics $F_{V^{(\bar{n},\bar{n})}|k}$ and $F_{V^{(\bar{n}-1:\bar{n})}|k}$ under two extreme scenarios: full independence or perfect correlation among private values. These two results imply that $F_{V^{(a:a)}|k}$ is bounded between $F^{+}_{V^{(a:a)}|k,a}$ and $F^{-}_{V^{(a:a)}|k,a}$ for all $a \leq \bar{n}-1$. Proposition 6 then follows from the fact that the ex ante surplus $\zeta_a(k)$ can be expressed as the difference between $\frac{1}{a+1}\int_{\underline{v}}^{\bar{v}} tdF_{V^{(a+1:a+1)}|k}(t)$ and $\frac{1}{a+1}\int_{\underline{v}}^{\bar{v}} tdF_{V^{(a:a+1)}|k}(t)$.

It then follows from Proposition 6 that a bidder's *ex ante* surplus prior to entry in the presence of n potential bidders (denoted by $\zeta^*(k,n)$) is bounded between $\zeta^*_L(k,n)$ and $\zeta^*_U(k,n)$ for all $n < \bar{n}$, where

$$\zeta^*_h(k,n) \equiv \textstyle\sum_{a=0}^{n-1} \zeta^h_a(k) \Pr(A_{-i} = a|k,n) \text{ for } h \in \{L,U\}. \tag{17}$$

Unlike in the case with independent private values where we could point identify $\zeta^*(k,n)$, here we can only recover a pair of bounds $\zeta^*_L(k,n)$ and $\zeta^*_U(k,n)$ when private values are affiliated. However, provided entry costs are observed from data, one can still test the chain of inequalities $\zeta^*_L(k,n) \leq k \leq \zeta^*_U(k,n)$ against the alternatives of "$k < \zeta^*_L(k,n)$" and "$k > \zeta^*_U(k,n)$", using a test statistic based on sample analogs and a bootstrap procedure. If the null is rejected in favor of one of the alternatives, we can conclude there is evidence in the data that supports hypotheses of risk-aversion or risk-loving. Unfortunately, on the other hand, a failure to reject the null does not necessarily allow us to conclude whether there is significant evidence for risk-neutrality or not.

The argument in the preceding paragraph can be extended where entry costs are measured with noise, and are orthogonal to the joint distribution of private values. In this case, bounds on ex ante surplus $\zeta^*(k,n)$ in (17) hold for all k given any n, except that the bounds on interim surplus $\{\zeta^h_a\}_{h=L,U}$ no longer depend on the unobserved cost k. Thus, similar to the case with IPV in Corollary 1, we can identify $E[\zeta^*_h(K,n)|n]$ as $\sum_{a=0}^{n-1} \zeta^h_a \Pr(A_{-i} = a|n)$ for $h = L, U$. Thus, provided the measurement errors are zero-mean so that $E[K]$ can be consistently estimated, we can construct a consistent test for the null "$E[\zeta^*_L(K,n)|n] \leq E[K] \leq E[\zeta^*_U(K,n)|n]$" against the alternatives "$E[K] < E[\zeta^*_L(K,n)|n]$" and "$E[K] > E[\zeta^*_U(K,n)|n]$". As before, a rejection of the null would provide statistically significant evidence again risk-neutrality while a failure to rejection would leave the test inconclusive.

6.2 Selective Entry with Informative Signals

Consider a model where each potential bidder observes in the entry stage a private signal correlated with his private value. Thus entry decisions depend on these informative signals. We identify bidders' risk attitudes in such a model, assuming the data report continuous variation in entry costs across auctions. As before, we fix N (the number of potential bidders) and suppress it from the notation. Auctions with endogenous and selective entry have been studied in Ye (2007), Gentry and Li (2013) and Marmer, Shneyerov and Xu (2013). Li, Lu and Zhao (2012) consider the same model of auctions with risk-averse bidders, affiliated signals and selective entry. Their emphasis is on the inference of the form of risk-aversions using data from both first-price and ascending auctions.

The model is specified as follows. In an entry stage, each potential bidder i observes a private signal S_i and decides whether to pay an entry cost k and enter the bidding stage. The entry cost is commonly known and the same for all bidders. Upon entry, each entrant draws a private value V_i, and competes in an ascending auction with a reserve price r. The joint distribution of $(S_1, ., S_N, V_1, ., V_N)$ as well as the constants k and r are common knowledge among potential bidders. Each entrant may or may not be aware of the number of active competitors A while bidding.

Assumption 6 *(i) (S_i, V_i) are identically and independently distributed across potential bidders; (ii) For each i, (S_i, V_i) are affiliated; (iii) Marginal distributions of S_i and V_i are continuous and increasing over bounded supports $[\underline{s}, \overline{s}]$ and $[\underline{v}, \overline{v}]$ respectively.*

Suppose the reserve price is binding with $r > \underline{v}$. Let $\overline{\omega}_i(s_i, k; s_{-i})$ denote ex ante utility from entry for bidder i with a signal s_i if potential competitors follow monotone, pure-strategy Bayesian Nash equilibria characterized by cutoffs $s_{-i} \equiv (s_j)_{j \neq i}$. That is, $\overline{\omega}_i(s_i, k; s_{-i}) \equiv E[u((V_i - P_i)_+ - k) \mid S_i = s_i, A_{-i} = \{j \neq i : S_j \geq s_j\}]$.

Lemma 3 *Under Assumption 6, $\overline{\omega}_i(s_i, k; s_{-i})$ is increasing in s_i and non-decreasing in s_{-i} for any k.*

Let $\overline{\omega}(s, k)$ be a shorthand for $\overline{\omega}_i(s, k; (s, ., s)) = E[u((V_i - P_i)_+ - k) \mid S_i = s, A_{-i} = \{j \neq i : S_j \geq s\}]$. There is no subscript i for $\overline{\omega}(s, k)$ because under Assumption 6, $\overline{\omega}(s, k)$ is the same for all i, and is increasing in s due to Lemma 3. An argument similar to that in Lemma 1 shows that under Assumption 6, there exists the following unique pure-strategy BNE in the entry stage: A potential bidder i enters iff $s_i \geq s_k^*$, where s_k^* solves $\overline{\omega}(s_k^*, k) = u(0)$ if $\overline{\omega}(\underline{s}, k) \leq u(0) \leq \overline{\omega}(\overline{s}, k)$. Otherwise, $s_k^* = \underline{s}$ (or $s_k^* = \overline{s}$) if $\overline{\omega}(\underline{s}, k) > u(0)$ (or $\overline{\omega}(\overline{s}, k) < u(0)$ respectively). Entry decisions in this model are selective in that the distribution of private values conditional on entry differs from the unconditional distributions.

To find bidders' risk attitudes when there is selective entry, we again exploit the fact that concave utilities lead to positive risk premia in the entry decisions. Let $\overline{\pi}(s, k) \equiv$

$E[(V_i - P_i)_+ - k \mid S_i = s, A_{-i} = \{j \neq i : S_j \geq s\}]$, which is identical for all i under Assumption 6. By the same argument as in Section 4.1, we can construct a test for bidders' risk attitudes as long as $\overline{\pi}(s_k^*, k)$ is identified from the distribution of transaction prices and entry decisions for any k that yields non-degenerate entry probabilities in equilibrium. Specifically, consider some entry cost k with $\underline{s} < s_k^* < \overline{s}$ so that uninteresting cases involving degenerate entry decisions are ruled out. (Whether a level of entry cost k leads to any non-degenerate entry probabilities is testable using the distribution of entry decisions from data.) Then, under Assumption 6, $\overline{\pi}(s_k^*, k) = 0$ when bidders are risk-neutral; and $\overline{\pi}(s_k^*, k) > 0$, or < 0, when bidders are risk-averse, or respectively risk-loving.

The test for risk attitudes introduced in Section 4.2 is not directly applicable under selective entry, because it leads to distorted decisions in favor of risk-aversion even when the true data-generating process has risk-neutral bidders. Such distortion arises from the affiliation between the private value V_i and the entry signal S_i. To see this, recall from the preceding paragraph that, with selective entry, bidders' risk attitudes are linked only to the sign of risk-premia for a marginal bidder whose signal equals the equilibrium cutoff s_k^*. If we were to apply the test from Section 4.2 in the presence of selective entry, it would amount to testing the sign of ex ante payoffs for a generic entrant whose signal is not restricted to be equal to the equilibrium cutoff (that is, $\tilde{\pi}(s_k^*, k) \equiv E[(V_i - P_i)_+ - k \mid S_i \geq s_k^*, A_{-i} \equiv \{j \neq i : S_j \geq s_k^*\}])$. However, the affiliation between S_i and V_i suggests ex ante payoffs for other *non-marginal* entrants must be greater than the marginal entrant's risk premium $\overline{\pi}(s_k^*, k)$. That is, $\tilde{\pi}(s_k^*, k)$ is bounded below by $\overline{\pi}(s_k^*, k)$. Consequently, applying the test from Section 4.2 for models with selective entry leads to over-rejection of risk-neutrality in favor of risk-aversion.[18]

The next proposition shows how to recover $\overline{\pi}(s_k^*, k)$ from entry probabilities and the distribution of transaction prices. To do this, it suffices to recover the distribution of V_i given $S_i = s_k^*$ and the distribution of P_i given $A_{-i} = \{j \neq i : S_j \geq s_k^*\}$ from the data.

Proposition 7 *Suppose (i) Assumption 6 holds with $(V_i, S_i)_{i \in N}$ independent of entry costs; (ii) for some fixed k, there exists $\varepsilon > 0$ such that $\underline{s} < s_{k'}^* < \overline{s}$ for all $k' \in (k - \varepsilon, k + \varepsilon)$; and (iii) entry costs are observed in the data. Then $\overline{\pi}(s_k^*, k)$ is identified from the distribution of transaction prices and entry probabilities.*

Implementing a test based on Proposition 7 requires the data to report continuous variation in the entry costs. As discussed in Section 4, such an assumption is strong and has limitations. It could fail under various empirical environments (say due to bias in the measurement of entry costs). It remains an open question whether additional exogenous variation could be used to construct a test under affiliated endogenous entry without requiring such

[18] We are grateful to a referee for this insight.

knowledge. We also leave the definition of a test statistic based on Proposition 7 and its asymptotic properties to future research.

7 Conclusion

We propose two tests for bidders' risk attitudes in ascending auctions where potential bidders make endogenous entry decisions based on their information of entry costs and ex ante knowledge of the distribution of private values. First, we show the risk premium can be non-parametrically recovered from the distribution of transaction prices and entry decisions, as long as the expected entry cost is identified from the data. Second, we show how exogenous variation in the number of potential bidders and observed auction heterogeneity provide additional identifying power to infer risk attitudes even when knowledge about entry costs is not available. Monte Carlo simulations suggest the test has reasonable finite-sample performance. Finally, we extend our results to identify bidders' risk attitudes in models with selective entry. Discussions about extensions to cases with affiliated private values are also provided.

There are several directions for future research. First, is it possible to recover the utility function over its full domain, not just the sign of risk attitudes? We conjecture this is possible with additional variation in entry costs or auction heterogeneity from the data. Some additional restrictions on how these factors affect private values may be needed for this purpose. Second, can we extend the analyses herein to ascending auctions with discrete increments? Haile and Tamer (2003) show how to form sharp bounds on value distributions using price distributions in this case. These bounds could in turn lead to bounds on the expectation of risk premia. Thus a test could be constructed to test the null that zero falls between these bounds. Similar to the case with affiliated private values in Section 6.1, rejection of the null could lead to a decision on risk attitudes while failure to reject the null leaves the test inconclusive. It remains an open question how to construct a more informative test under additional economic restrictions on the model elements.

APPENDIX

A Proofs of Identification

Proof of Lemma 1. First, we show that under Assumption 1, $\omega(k;\lambda_{-i})$ is continuous and decreasing in λ_{-i} for all k. To see this, recall by the Law of Iterated Expectations,

$$\omega(k;\boldsymbol{\lambda}_{-i}) = u(-k)F_{V|k}(r) + \int_r^{\bar{v}} h(v,k;\boldsymbol{\lambda}_{-i})dF_{V|k}(v) \tag{A1}$$

where for all $v > r$,

$$\begin{aligned} h(v,k;\boldsymbol{\lambda}_{-i}) \;\; &\equiv \;\; u(v-r-k)F_{P_i}(r \mid k,\boldsymbol{\lambda}_{-i}) + \int_r^v u(v-p-k)dF_{P_i}(p \mid k,\boldsymbol{\lambda}_{-i}) \\ + \;\; & \;\; u(-k)[1-F_{Pi}(v \mid k,\boldsymbol{\lambda}_{-i})] \end{aligned}$$

and we have used the independence between private values V_i conditional on entry costs. Note for any $t \in [r,\,\bar{v}]$, the event "$P_i \leq t$" can be represented as

$$\cap_{j\in N\setminus\{i\}}\{\text{"}j \textit{ stays out}\text{" or "}j \textit{ enters} \cap V_j \leq t\text{"}\}.$$

Due to the independence between entry decisions and between private values, $F_{P_i}(t \mid k,\boldsymbol{\lambda}_{-i}) = \prod_{j\neq i}[1-\lambda_j+\lambda_j F_{V|k}(t)]$. The marginal effect of λ_j on this conditional probability is strictly negative for all $j \neq i$ at $\lambda_j \in [0,1]$ and $t \in [r,\bar{v})$. (Recall r is a binding reserve price.) This implies $h(v,k;\boldsymbol{\lambda}_{-i})$ is decreasing in $\boldsymbol{\lambda}_{-i}$ given any k and any $v \in [r,\bar{v}]$. Hence $\omega(k;\boldsymbol{\lambda}_{-i})$ is decreasing in $\boldsymbol{\lambda}_{-i}$. Continuity follows from an application of the Dominated Convergence Theorem. The rest of the proof is similar to the risk-neutral case in Levin and Smith (1994) and omitted. □

Proof of Proposition 1. Conditional on k, entry decisions are independent across bidders, and jointly independent of private values. Besides, private values are i.i.d. across bidders given k. Hence, once conditional on k and the realization of A_{-i}, (V_i, P_i) are independent of mixed strategies adopted by potential competitors. Thus

$$E[(V_i-P_i)_+ \mid k] = \textstyle\sum_{a=0}^{N-1} E[(V_i-P_i)_+ \mid k,\; A_{-i}=a]\Pr(A_{-i}=a \mid k). \tag{A2}$$

With k and entry decisions observed from the data, λ_k^* is directly identified as the probability that a bidder enters under the entry cost k. Consequently, $\Pr(A_{-i}=a \mid k)$ can be recovered as a binomial distribution with parameters $N-1$ and λ_k^*. Conditional on entering with the cost k, private values are independent draws from $F_{V|k}$. Let $1\{.\}$ denote the indicator function. By the Law of Iterated Expectations, $E[(V_i-P_i)_+ \mid k,\,A_{-i}=a]$ is

$$\begin{aligned} & E[(V_i-P_i)1\{V_i>P_i>r\} \mid k,\,A_{-i}=a] + E[(V_i-r)1\{V_i>r\}1\{P_i=r\} \mid k,\,A_{-i}=a] \\ = \;\; & \int_r^{\bar{v}}\left(\int_v^{\bar{v}}(s-v)\,dF_{V|k}(s)\right)dF_{V|k}(v)^a + F_{V|k}(r)^a\int_r^{\bar{v}}(v-r)\,dF_{V|k}(v) \end{aligned} \tag{A3}$$

for all $a \geq 1$, due to independence between V_i and P_i given k. Applying integration by parts to the first term on the right-hand side of (A3), we have

$$E[(V_i - P_i)_+ \mid A_{-i} = a, k] = \int_r^{\bar{v}} \left(F_{V|k}(v)^a - F_{V|k}(v)^{a+1} \right) dv$$

for $a \geq 1$. Besides, $E[(V_i - P_i)_+ \mid A_{-i} = 0, k] = E[(V_i - r)_+ \mid k] = \int_r^{\bar{v}} \left[1 - F_{V|k}(v)\right] dv$.

With k assumed known from the data, $\pi(k)$ is recoverable as long as $F_{V|k}(v)$ is identified for $v \geq r$. Let W denote transaction prices observed in data. If no entrants bid above r, then define $W < r$. The symmetric IPV assumption implies for any $m \geq 2$, $\Pr(W < r \mid A = m, k) = \Pr(V^{(m:m)} < r \mid k) = F_{V|k}(r)^m$ and $\Pr(W = r \mid A = m, k) = mF_{V|k}(r)^{m-1}[1 - F_{V|k}(r)]$. Hence for any $m \geq 2$ and $t \geq r$,

$$\begin{aligned}
& \Pr(W \leq t \mid A = m, k) \\
= \; & \Pr(W < r \mid A = m, k) + \Pr(W = r \mid A = m, k) + \Pr(r < W \leq t \mid A = m, k) \\
= \; & F_{V|k}(t)^m + mF_{V|k}(t)^{m-1}[1 - F_{V|k}(t)] = F_{V^{(m-1:m)}|k}(t).
\end{aligned}$$

For any $m \geq 2$, define $\phi_m(t) \equiv t^m + mt^{m-1}(1-t)$ so that $F_{V^{(m-1:m)}|k}(t) = \phi_m(F_{V|k}(t))$. Since $\phi_m(t)$ is one-to-one for any $m \geq 2$ over $t \in [0,1]$, $F_{V|k}(t)$ is (over-)identified for each $t \geq r$ from the distributions of W conditional on k and $A = m$. $\square$

Proof of Corollary 1. *Part (a).* By Proposition 1 and the condition in part (a), $\pi(k) = 0$ for all $k \in [\underline{k}, \overline{k}]$ if bidders are risk-neutral, and $\pi(k) > 0$ (or $\pi(k) < 0$) for all k if bidders are risk-averse (or risk-loving). Integrating out k using F_K proves (a).

Part (b). Independence between K and $(V_i)_{i \in N}$ implies that, given $A_{-i} = a$, the vector of ordered values $\{V^{(s:a+1)}\}_{s=1}^{a+1}$ is independent of K. Thus $\zeta_a \equiv E[(V_i - P_i)_+ \mid A_{-i} = a, k]$ does not depend on k for all $a \geq 0$ (recall $P_i = r$ when $A_{-i} = \varnothing$). By the Law of Total Probability, $E[\pi(K)]$ is:

$$\int_{\underline{k}}^{\overline{k}} \left(-k + \textstyle\sum_{a=0}^{N-1} \zeta_a \Pr(A_{-i} = a \mid k) \right) dF_K(k) = \textstyle\sum_{a=0}^{N-1} \zeta_a \Pr(A_{-i} = a) - \mu_K. \tag{A4}$$

To identify $\Pr(A_{-i} = a)$ (or $\int_{\underline{k}}^{\overline{k}} \Pr(A_{-i} = a \mid k) dF_K$), note that given any k and N, A_{-i} is binomial $(N-1, \lambda_k^*)$ while A is binomial (N, λ_k^*). By construction,

$$\Pr(A_{-i} = a \mid k) = \tfrac{N-a}{N} \Pr(A = a \mid k) + \tfrac{a+1}{N} \Pr(A = a+1 \mid k) \tag{A5}$$

for all k and $0 \leq a \leq N-1$. Integrating out k on both sides of (A5) implies $\Pr(A_{-i} = a) = \frac{N-a}{N} \Pr(A = a) + \frac{a+1}{N} \Pr(A = a+1)$. Since the unconditional distribution of A is directly identified, so is the distribution of A_{-i}.[19] With μ_K assumed known, this implies $E[\pi(K)]$ is identified. $\square$

[19]In fact this provides us with a set of testable implications that can be used for testing the model specification of symmetric independent private values.

Proof of Lemma 2. First note $\psi_1^* \neq u(0)$ under Assumption 3 and 4. [To see this, consider the case with z and $N = 2$. Then under Assumption 4, $\rho_{0,z,2}\psi_0(z) + \rho_{1,z,2}\psi_1^* = u(0)$ with $\rho_{0,z,2} > 0$, $\rho_{1,z,2} > 0$ and $\rho_{0,z,2} + \rho_{1,z,2} = 1$. It follows from Assumption 3 that

$$\begin{aligned} & \psi_0(z) \equiv E[u(V_i - k) \mid Z = z] \\ > \quad & \psi_1(z) \equiv \psi_1^* = E[u((V_i - V_j)_+ - k) \mid Z = z]. \end{aligned}$$

This implies $\psi_1^* < u(0)$.] By construction, the r-th row of the coefficient matrix on the left-hand side of (12) and the r-th element of the coefficient vector in front of ψ_1^* on the right-hand side of (12) add up to 1. Hence we can write (12) as

$$\overline{\mathbf{M}} \cdot [(\psi_0(z''), \psi_0(z'), \psi_0(z), \psi_2^*, ..., \psi_{n-1}^*)' - u(0)] = -\overline{\boldsymbol{\rho}_1} \cdot [\psi_1^* - u(0)]\,. \tag{A6}$$

where $\overline{\mathbf{M}}$ denotes the $(n+1)$-by-$(n+1)$ matrix of coefficients on the left-hand side of (12); and $\overline{\boldsymbol{\rho}_1}$ denotes the $(n+1)$-by-1 vector of coefficients in front of ψ_1^* on the right-hand side of (12). The matrix of coefficients $\overline{\mathbf{M}}$ has full-rank $n+1$. To see this, note the determinant of $\overline{\mathbf{M}}$ is just the product of its diagonal elements which are all positive under Assumption 4.[20] Hence with $\psi_1^* \neq u(0)$ and $\overline{\mathbf{M}}$ non-singular, (A6) can be rescaled and written as

$$\overline{\mathbf{M}} \cdot (\tilde{\psi}_{0,z''}, \tilde{\psi}_{0,z'}, \tilde{\psi}_{0,z}, \tilde{\psi}_2, .., \tilde{\psi}_{n-1})' = -\overline{\boldsymbol{\rho}_1} \cdot c \tag{A7}$$

where

$$(\tilde{\psi}_{0,z''}, \tilde{\psi}_{0,z'}, \tilde{\psi}_{0,z}, \tilde{\psi}_2, .., \tilde{\psi}_{n-1})' \equiv \frac{c}{\psi_1^* - u(0)}[(\psi_0(z''), \psi_0(z'), \psi_0(z), \psi_2^*, .., \psi_{n-1}^*)' - u(0)].$$

The claim of the lemma then follows immediately. $\square$

Proof of Proposition 5. *Part (a).* It suffices to note that, under Assumption 3, $E(W|Z = z, A = a) = h(z) + E[\eta^{(a-1:a)}]$ for all z and $a \geq 2$, and $E[\eta^{(a-1:a)}]$ is independent of z. Hence the claim in (a) holds.

Part (b). By construction,

$$\psi_0(z) \equiv E[u(V_i - k) \mid Z = z] = E[u(h(z) + \eta_i - k)]$$

where the first expectation is taken with respect to V_i given $Z = z$ and the second is with respect to η_i alone while z is fixed at a realized value. By Lemma 2, $(\tilde{\psi}_{0,z'} - \tilde{\psi}_{0,z})/(\tilde{\psi}_{0,z''} - \tilde{\psi}_{0,z'})$ equals $[\psi_0(z') - \psi_0(z)]/[\psi_0(z'') - \psi_0(z')]$, which in turn equals:

$$\frac{E[u(h(z') + \eta_i - k)] - E[u(h(z) + \eta_i - k)]}{E[u(h(z'') + \eta_i - k)] - E[u(h(z') + \eta_i - k)]}. \tag{A8}$$

[20] We thank an anonymous referee for pointing this out.

By the independence between Z and η in Assumption 3 and the Mean Value Theorem, (A8) equals:

$$\frac{[h(z') - h(z)]\, E[u'(\alpha h(z') + (1-\alpha)h(z) + \eta_i - k)]}{[h(z'') - h(z')]E[u'(\alpha' h(z'') + (1-\alpha')h(z') + \eta_i - k)]} \tag{A9}$$

for some α and $\alpha' \in (0,1)$ that depend on (z, z') and (z', z'') respectively. By the ordering of $h(z'')$, $h(z')$ and $h(z)$, it must be the case that

$$\alpha h(z') + (1-\alpha)h(z) + \eta_i - k < \alpha' h(z'') + (1-\alpha')h(z') + \eta_i - k$$

for any k, η_i, α and α'.

If bidders are risk-neutral, then

$$u'(\alpha h(z') + (1-\alpha)h(z) + \eta_i - k) = u'(\alpha' h(z'') + (1-\alpha')h(z') + \eta_i - k) > 0 \tag{A10}$$

for all k, η_i, α and α', and (A9) equals $[h(z') - h(z)]/[h(z'') - h(z')]$. Otherwise if bidders are risk-averse then the equality in (A10) is replaced by a strict inequality "$>$" for all k, η_i, α and α' due to the concavity of u. It then follows from the independence between η_i and Z that

$$\frac{E[u'(\alpha h(z') + (1-\alpha)h(z) + \eta_i - k)]}{E[u'(\alpha' h(z'') + (1-\alpha')h(z') + \eta_i - k)]} > 1$$

when bidders are risk-averse. Therefore (A9) is strictly greater than $[h(z') - h(z)]/[h(z'') - h(z')]$ under risk aversion. A symmetric argument shows the reverse strict inequality holds when bidders are risk-loving. $\square$

Lemma A1 *Let the joint distribution of private values $F_{(V_i)_{i=1}^n|k}$ be affiliated and exchangeable for any number of potential bidders n and any entry cost k. If $\omega(k;(1,.,1)) < u(0) < \omega(k;(0,.,0))$ where ω is defined as in (1), then there exists a unique symmetric BNE in which all bidders enter independently with probability λ_k^*, where λ_k^* solves $\omega(k;(\lambda_k^*,.,\lambda_k^*)) = u(0)$.*

Proof of Lemma A1. It suffices to show $F_{P_i}(t \mid k, \boldsymbol{\lambda}_{-i})$ is decreasing in $\boldsymbol{\lambda}_{-i}$ over $(\underline{v}, \overline{v})$ given k and n. To see this, note by the Law of Total Probability,

$$F_{P_i}(t|k, \boldsymbol{\lambda}_{-i}) \equiv \textstyle\sum_{a=0}^{n-1} F_{V^{(a:a)}|k}(t) \Pr(A_{-i} = a \mid \boldsymbol{\lambda}_{-i}, k),$$

where $F_{V^{(a:a)}|k}(t)$ denotes the distribution of the largest order statistic from a a-dimensional random vector $(V_i)_{i \le a}$; and define $F_{V^{(0:0)}|k}(t) \equiv 1$ for $t \ge \underline{v}$. Under the assumptions of the lemma, $F_{V^{(a':a')}|k}$ first-order stochastically dominates $F_{V^{(a:a)}|k}$ if $a' > a$. Besides, $\Pr(A_{-i} = a \mid \boldsymbol{\lambda}_{-i}, k)$ is stochastically increasing in $\boldsymbol{\lambda}_{-i}$ since bidders enter independently with respective probabilities. Thus for all $t \in (\underline{v}, \overline{v})$, $F_{P_i}(t \mid k, \boldsymbol{\lambda}_{-i})$ is decreasing in $\boldsymbol{\lambda}_{-i}$ given k and n. The rest of the proof follows from the same argument used to prove Lemma 1. $\square$

Proof of Lemma 3. Let $P_i \equiv \max_{j\in A\backslash\{i\}}\{\max\{V_j, r\}\}$, and let $F_{P_i|v_i,s_i,s_{-i}}$ denote the C.D.F. of P_i conditional on $(V_i, S_i, S_{-i}) \equiv (v_i, s_i, s_{-i})$. Likewise define $F_{V_i|s_i,s_{-i}}$ as the distribution of V_i conditional on $(S_i, S_{-i}) = (s_i, s_{-i})$. Under Assumption 6, $F_{P_i|v_i,s_i,s_{-i}}(p_i) = F_{P_i|s_{-i}}(p_i)$ and $F_{V_i|s_i,s_{-i}}(v_i) = F_{V_i|s_i}(v_i)$ for any (v_i, p_i, s_i, s_{-i}). Thus by the Law of Iterated Expectations,

$$\bar{\omega}_i(s_i, k; s_{-i}) = u(-k)F_{V_i|s_i}(r) + \int_r^{\overline{v}} \bar{h}(v, k; s_{-i})dF_{V_i|s_i}(v)$$

where $\bar{h}(v, k; s_{-i}) \equiv u(v-r-k)F_{P_i|s_{-i}}(r)+ \int_r^v u(v-p-k)dF_{P_i|s_{-i}}(p)+ u(-k)[1-F_{P_i|s_{-i}}(v)]$. By the Leibniz Rule,

$$\frac{\partial}{\partial v}\bar{h}(v, k; s_{-i}) = u'(v-r-k)F_{P_i|s_{-i}}(r) + \int_r^v u'(v-p-k)dF_{P_i|s_{-i}}(p) > 0 \tag{A11}$$

Thus $\bar{h}$ is increasing in v for any fixed s_{-i} and k. By the affiliation of V_i and S_i for all i, the distribution $F_{V_i|s_i}(.)$ is stochastically increasing in s_i. Hence $\bar{\omega}_i(s_i, k; s_{-i})$ is increasing in s_i given k and s_{-i}.

To show $\bar{\omega}_i(s_i, k; s_{-i})$ is non-decreasing in s_{-i} given s_i and k, it suffices to show $F_{P_i|s_{-i}}(p)$ is stochastically non-decreasing in s_{-i} for all $p \geq r$, which would imply $\bar{h}(v, k; s_{-i})$ is non-decreasing in s_{-i} for any $v \in [\underline{v}, \bar{v}]$. Note for any $t \in [r, \bar{v}]$, the event "$P_i \leq t$" can be written as

$$\cap_{j\neq i}\{"S_j < s_j" \cup "S_j \geq s_j \cap V_j \leq t"\}$$

Under Assumption 6, $\Pr(P_i \leq t \mid s_{-i}) = \prod_{j\neq i}[F_{S_j}(s_j) + \Pr(V_j \leq t, S_j \geq s_j)]$. Also note for all t and any $s_j' > s_j$,

$$F_{S_j}(s_j) + \Pr(V_j \leq t, S_j \geq s_j) \leq F_{S_j}(s_j') + \Pr(V_j \leq t, S_j \geq s_j').$$

Hence $F_{P_i|S_{-i}}(t \mid s_{-i})$ is non-decreasing in s_{-i} for all $t \in [r, \bar{v}]$. $\square$

Lemma A2 *Under Assumption 6, for any k such that $\underline{s} < s_k^* < \overline{s}$, $F_{V_i|S_i\geq s_k^*}(t)$ is identified for $t \geq r$ from the distribution of W conditional on any k and any number of entrants a (with $a \geq 2$).*

Proof of Lemma A2. By definition, for any $t \geq r$, $\Pr(W \leq t \mid k, A = a)$ is identical to the distribution of the second-highest order statistic among a independent draws from the same conditional distribution $F_{V_i|S_i\geq s_k^*}$. That is, for any $t \geq r$

$$\Pr(W \leq t \mid k, A = a) = \textstyle\sum_{m=a-1}^{a} \binom{a}{m} F_{V_i|S_i\geq s_k^*}(t)^m[1-F_{V_i|S_i\geq s_k^*}(t)]^{a-m}$$

Thus for all $a \geq 2$, there exists a one-to-one mapping ϕ_a so that $F_{V_i|S_i\geq s_k^*}(t) = \phi_a^{-1}(\Pr(W \leq t \mid k, A = a))$. Note the mapping ϕ_a does not depend on k as (V_i, S_i) is assumed to be independent of entry costs. That $\Pr(i\ enters|\ k) > 0$ implies $\Pr(A \geq 2 \mid k) > 0$. Thus $F_{V_i|S_i \geq s_k^*}(t)$ is over-identified for $t \geq r$, because the identification arguments above can be applied for any a such that $\Pr(A = a \mid k) > 0$. $\square$

Lemma A3 *Under Assumption 6, $F_{V_i|S_i=s_k^*}(t)$ is identified for any $t \geq r$ and any k s.t. $\underline{s} < s_k^* < \overline{s}$, provided* $\Pr(W \leq t \mid k', A = a)$ *and* $\Pr(i\ enters|\ k')$ *are identified for all k' in an open neighborhood around k for some $a \geq 2$.*

Proof of Lemma A3. Given Lemma A2, $F_{V_i|S_i \geq s_k^*}(t)$ is identified for any such k using the distribution of entry decisions and transaction prices. Hence

$$\begin{aligned} \Pr(V_i \leq t, S_i \geq s_k^*) &= F_{V|S_i \geq s_k^*}(t) \Pr(i\ enters \mid k) \\ &= \phi_a^{-1}(\Pr(W \leq t \mid k, A = a)) \Pr(i\ enters \mid k) \end{aligned} \tag{A12}$$

is also identified using any a such that $\Pr(A = a \mid k) > 0$. We consider this joint distribution as known for the rest of the proof. For any $t \geq r$, differentiating this distribution with respect to entry costs at k gives:

$$\begin{aligned} & \tfrac{\partial}{\partial K} \Pr(V_i \leq t, S_i \geq s_K^*)|_{K=k} \\ = \; & -\tfrac{\partial}{\partial K} \Pr(V_i \leq t, S_i \leq s_K^*)|_{K=k} = -\left[\tfrac{\partial}{\partial S} \Pr(V_i \leq t, S_i \leq S)|_{S=s_k^*}\right] \left(\tfrac{ds_K^*}{dK}|_{K=k}\right) \\ = \; & -F_{V|S=s_k^*}(t) f_S(s_k^*) \left(\tfrac{ds_K^*}{dK}|_{K=k}\right) = -F_{V_i|S_i=s_k^*}(t) \left(\tfrac{dF_S(s_K^*)}{dK}|_{K=k}\right), \end{aligned}$$

where we have used the independence between (V_i, S_i) and entry costs. [Gentry and Li (2013) used this derivative-based argument in their derivation for bounds on value distributions in auction models with endogenous entry.] Hence

$$F_{V_i|S_i=s_k^*}(t) = -\frac{\frac{\partial}{\partial K} \Pr(V_i \leq t, i\ enters \mid K)|_{K=k}}{\frac{d}{dK} \Pr(i\ does\ not\ enter \mid K)|_{K=k}}$$

because $F_S(s_k^*) = \Pr(i\ does\ not\ enter|\ k)$ in the pure-strategy BNE and $\frac{d}{dK} F_S(s_K^*) = \frac{d}{dK} \Pr(i$ *does not enter*$|\ K)$. The denominator is non-zero under the assumption of the proposition. Hence $F_{V_i|S_i=s_k^*}(t)$ is identified for $t \geq r$ as long as $\Pr(W \leq t \mid k', A = a)$ and $\Pr(i\ enters|\ k')$ are identified for all k' in an open neighborhood around k for some $a \geq 2$. $\square$

Proof of Proposition 7. Because (V_i, S_i) are independent across bidders, the joint distribution of (V_i, P_i) conditional on $A_{-i} = \{j \neq i : S_j \geq s_k^*\}$ and $S_i = s_k^*$, evaluated at $(P_i, V_i) = (p, v)$, are factored as:

$$F_{P_i|A_{-i}=\{j \neq i : S_j \geq s_k^*\}}(p) F_{V_i|S_i=s_k^*}(v). \tag{A13}$$

To identify $\overline{\pi}$, it suffices to recover the two conditional distributions in (A13) at all p and $v \geq r$. Because (V_i, S_i) are identically distributed across i, for any $t \geq r$,

$$\begin{aligned} & \Pr(P_i \leq t \mid A_{-i} = \{j \neq i : S_j \geq s_k^*\}) \\ = \; & \textstyle\sum_{m=0}^{N-1} \left[F_{V|S \geq s_k^*}(t)\right]^m \binom{N-1}{m} F_S(s_k^*)^{N-m-1}[1 - F_S(s_k^*)]^m \end{aligned} \tag{A14}$$

where N is the number of potential bidders (including i). The subscript i is suppressed to simplify the notation. By construction, $F_{S_i}(s_k^*) = \Pr(i \textit{ enters} \mid k)$ is identified. Let a denote the realized number of entrants. First, under Assumption 6, $F_{V_i|S_i \geq s_k^*}(t)$ is over-identified for $t \geq r$ from the distribution of prices W under k and *any* $a \geq 2$. (See Lemma A2 above for details.) Next, Lemma A3 shows that, under Assumption 6, $F_{V_i|S_i = s_k^*}(t)$ is identified for any $t \geq r$ provided that for some $a \geq 2$, $\Pr(W \leq t \mid k', A = a)$ and $\Pr(i \text{ enters} | \; k')$ are observed for all k' in an open neighborhood around k. Thus the conditional distributions in (A13), as well as $\overline{\pi}(s_k^*)$, are identified. $\square$

B Limiting Distribution of the Test Statistic

Let $F_{W|A=m}$ denote the distribution of W given $A = m$; and let $\mathbf{F}_W \equiv (F_{W|A=2}, F_{W|A=3}, ..., F_{W|A=N})$. Let $\mathcal{B}_{[\underline{v},\overline{v}]}$ denote the space of bounded functions with a domain $[\underline{v}, \overline{v}]$; and let $\mathcal{D}_{[\underline{v},\overline{v}]} \subset \mathcal{B}_{[\underline{v},\overline{v}]}$ denote the space of non-decreasing functions that are right-continuous with left limits and map from $[\underline{v}, \overline{v}]$ into $[0, 1]$. For any $\mathbf{F}_W$ such that $F_{W|A=m} \in \mathcal{D}_{[\underline{v},\overline{v}]}$ for all $2 \leq m \leq N$, let $\zeta(\mathbf{F}_W) \equiv (\zeta_0(\mathbf{F}_W), \zeta_1(\mathbf{F}_W), ..., \zeta_{N-1}(\mathbf{F}_W))'$ where

$$\zeta_a(\mathbf{F}_W) \equiv \int_r^{\overline{v}} \left\{ \left[\tfrac{1}{N-1} \textstyle\sum_{m=2}^{N} \phi_m^{-1}(F_{W|A=m}(s))\right]^a - \left[\tfrac{1}{N-1} \textstyle\sum_{m=2}^{N} \phi_m^{-1}(F_{W|A=m}(s))\right]^{a+1} \right\} ds \tag{B15}$$

for all $0 \leq a \leq N-1$ and $2 \leq m \leq N$. Let $\hat{F}_{W|A=m,T}$ denote estimators for $F_{W|A=m}$ defined in Section 4.2, and let $\hat{\mathbf{F}}_{W,T} \equiv (\hat{F}_{W|A=m,T} : 2 \leq m \leq N)$. Define $\hat{\zeta}_T \equiv (\hat{\zeta}_{0,T}, \hat{\zeta}_{1,T}, \hat{\zeta}_{2,T}, ..., \hat{\zeta}_{N-1,T})' = \zeta(\hat{\mathbf{F}}_{W,T})$.

Throughout this section, let "$\to$" denote the convergence of sequences in $\mathcal{B}_{[r,\overline{v}]}$ in terms of the uniform metric; and let "$\rightsquigarrow$" denote the weak convergence of stochastic processes. For the product space $\mathcal{B}_{[\underline{v},\overline{v}]} \otimes \mathbb{R}^{N+1}$, let the metric be defined as the maximum of the uniform metric over $\mathcal{B}_{[\underline{v},\overline{v}]}$ and the Euclidean metric over $\mathbb{R}^{N+1}$.

To fix the main idea and simplify the presentation, we establish Proposition 3 for the cases where either $N = 2$ or the average in the square brackets in (B15) is replaced by a single estimate $\phi_m^{-1}(\hat{F}_{W|A=m,T}(s))$ for some $2 \leq m \leq N$. In other words, these are the cases where the definition of components in ζ in (B15) is simplified to:

$$\zeta_a(F_{W|A=m}) \equiv \textstyle\int_r^{\overline{v}} \left[\left(\phi_m^{-1}(F_{W|A=m}(s))\right)^a - \left(\phi_m^{-1}(F_{W|A=m}(s))\right)^{a+1} \right] ds \tag{B16}$$

for some $m \geq 2$. With a slight abuse of notation, we write ζ as a function of $F_{W|A=m}$ for $m \geq 2$. That is, $\zeta(F_{W|A=m}) \equiv (\zeta_0(F_{W|A=m}),\ \zeta_1(F_{W|A=m}),\ ...,\ \zeta_{N-1}(F_{W|A=m}))'$ accordingly, where $\zeta_a(F_{W|A=m})$ is defined in (B16). Establishing the limiting distribution in the general case with $N \geq 3$ in (B15) essentially requires the same type of arguments presented below.

For any $a \geq 1$, $m \geq 2$ and $\alpha \in [0,1]$, define $\chi_{a,m}(\alpha) \equiv [\phi_m^{-1}(\alpha)]^a$ so that

$$\chi'_{a,m}(\alpha) = a\left[\phi_m^{-1}(\alpha)\right]^{a-1}\left(d\phi_m^{-1}(t)/dt\Big|_{t=\alpha}\right).$$

Let $\hat{\mathcal{D}}_{[\underline{v},\overline{v}]}$ denote the subset of $\mathcal{D}_{[\underline{v},\overline{v}]}$ that consists of step functions only.

Lemma B4 *Suppose Assumptions 1 and 2 hold. For any* $1 \leq a \leq N$ *and* $2 \leq m \leq N$ *and any sequence* $F_{W|A=m,T} \in \hat{\mathcal{D}}_{[\underline{v},\overline{v}]}$ *such that* $\sqrt{T}\left(F_{W|A=m,T} - F_{W|A=m}\right) \to F^*$ *for some* $F^* \in \mathcal{B}_{[\underline{v},\overline{v}]}$,

$$\lim_{T\to\infty}\sqrt{T}\int_r^{\overline{v}}[\chi_{a,m}(F_{W|A=m,T}(w)) - \chi_{a,m}(F_{W|A=m}(w))]dw = \int_r^{\overline{v}}\chi'_{a,m}(F_{W|A=m}(w))F^*(w)dw. \tag{B17}$$

The property established in Lemma B4 is part of the conditions needed for applying the Extended Continuous Mapping Theorem (see Theorem 1.11.1 in van der Vaart and Wellner (1996), page 67) to derive the limiting distribution of our test statistic. This approach does *not* require the stochastic process$\sqrt{T}[\chi_{a,m}(\hat{F}_{W|A=m,T}) - \chi_{a,m}(F_{W|A=m})]$, indexed over $[\underline{v},\overline{v}]$, to converge weakly to a tight Gaussian process.[21] Proof of Lemma B4 is technical and presented in Section 2 of the web supplement Fang and Tang (2014).

For each $m \geq 2$, let $\mathbb{F}_{W|A=m}$ denote a zero-mean Gaussian Process indexed over $[\underline{v},\overline{v}]$ such that (a) for any $s < v$ on $[\underline{v},\overline{v}]$ the covariance between $\mathbb{F}_{W|A=m}(s)$ and $\mathbb{F}_{W|A=m}(v)$ is:

$$[\Pr(A=m)]^{-1}\begin{bmatrix} F_{W|A=m}(s)\left(1-F_{W|A=m}(s)\right) & F_{W|A=m}(s)\left(1-F_{W|A=m}(v)\right) \\ F_{W|A=m}(s)\left(1-F_{W|A=m}(v)\right) & F_{W|A=m}(v)\left(1-F_{W|A=m}(v)\right) \end{bmatrix}; \tag{B18}$$

and (b) the covariance between $\mathbb{F}_{W|A=m}(s)$ and $\mathbb{F}_{W|A=m'}(s')$ is 0 for any $s, s' \in [\underline{v},\overline{v}]$ and $m \neq m'$.

Let $(\mathcal{N}'_\rho, \mathcal{N}_\mu)'$ be a multivariate normal vector in $\mathbb{R}^{N+1}$ such that, for any finite subset $\{v_1, v_2, ..., v_J\} \subset [\underline{v},\overline{v}]$, the $(J+N+1)$-vector $(\mathbb{F}_{W|A=m}(v_1),\ ...,\ \mathbb{F}_{W|A=m}(v_J),\ \mathcal{N}'_\rho,\ \mathcal{N}_\mu)'$ has a multivariate normal distribution with a zero mean and a covariance matrix that depends on population moments in the following way: (i) The covariance between $\mathbb{F}_{W|A=m}(s)$ and $\mathbb{F}_{W|A=m}(v)$ for any s, v on $[\underline{v},\overline{v}]$ is given in (B18). (ii) The variance of the a-th element in $\mathcal{N}_\rho$, denoted $\mathcal{N}_{\rho,a}$, is $(\frac{N-a}{N})^2\Pr(A=a)+$ $(\frac{a+1}{N})^2\Pr(A=a+1)-$ $(\rho_a)^2$, where $\rho_a \equiv \Pr(A_{-i}=a)$. The covariance between $\mathcal{N}_{\rho,a}$ and $\mathcal{N}_{\rho,b}$ with $a < b$ is $\bar{\rho}_{a,b} - \rho_a\rho_b$, where $\bar{\rho}_{a,b} \equiv \frac{b(N-b)}{N^2}\Pr(A=b)$ if $b = a+1$ and $\bar{\rho}_{a,b} = 0$ otherwise. (iii) The variance of $\mathcal{N}_\mu$ is the sum of the variance of K

[21] We are grateful to the Associate Editor for pointing this out.

and the variance of ϵ. The covariance between $\mathcal{N}_\mu$ and $\mathcal{N}_{\rho,a}$ is $\frac{N-a}{N}\mu_a + \frac{a+1}{N}\mu_{a+1} - \rho_a E(K)$, where $\mu_a \equiv E[K1\{A = a\}]$. (iv) For any $v \in [\underline{v}, \overline{v}]$, the covariance between $\mathbb{F}_{W|A=m}(v)$ and $\mathcal{N}_\mu$ is $[\Pr(A = m)]^{-1} \left[\mu_{v,m} - \mu_m F_{W|A=m}(v)\right]$, where $\mu_{v,m} \equiv E[1\{W \leq v, A = m\}K]$. (v) For any a and v, the covariance between $\mathbb{F}_{W|A=m}(v)$ and $\mathcal{N}_{\rho,a}$ is given by the off-diagonal entry of the 2-by-2 matrix $\tilde{D}\tilde{\Sigma}\tilde{D}'$, where $\tilde{\Sigma}$ is the 4-by-4 covariance matrix of $[1\{W \leq v, A = m\}$, $1\{A = m\}$, $1\{A = a\}$, $1\{A = a + 1\}]'$ if $m > a + 1$ or $m < a$; or $\tilde{\Sigma}$ is the 3-by-3 covariance matrix of $[1\{W \leq v, A = m\}$, $1\{A = a\}$, $1\{A = a + 1\}]'$ otherwise. The matrix $\tilde{D}$ is the 2-by-4 (or 2-by-3) Jacobian matrix needed to apply the multivariate Delta Method. Under Assumptions 1 and 2, both $\tilde{D}$ and $\tilde{\Sigma}$ have full rank.

Lemma B5 *Suppose Assumptions 1 and 2 hold. For any $m \geq 2$,*

$$\sqrt{T}\begin{pmatrix} \hat{F}_{W|A=m,T} - F_{W|A=m} \\ \hat{\rho}_T - \rho \\ \frac{1}{T}\sum_t \tilde{K}_t - E(K) \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathbb{F}_{W|A=m} \\ \mathcal{N}_\rho \\ \mathcal{N}_\mu \end{pmatrix}. \tag{B19}$$

Lemma B5 follows from a standard argument that uses the Donsker property of classes of indicator functions of lower rectangles and equalities, as well as the Donsker property of their pairwise minimum and finite unions. Its proof follows from standard arguments and is omitted for brevity. We present details in the derivation of the covariance between $\mathbb{F}_{W|A=m}$, $\mathcal{N}_\rho$ and $\mathcal{N}_u$ in the web supplement Fang and Tang (2014).

Let $\hat{\mathcal{D}}_T$ denote $\{b \in \mathcal{B}_{[\underline{v},\overline{v}]} : F_{W|A=m} + b/\sqrt{T} \in \hat{\mathcal{D}}_{[\underline{v},\overline{v}]}\}$. Define $g_T : \hat{\mathcal{D}}_T \otimes \mathbb{R}^{N+1} \to \mathbb{R}^{2N+1}$ as

$$g_T(b, c_1, c_2) \equiv \left(\sqrt{T}\left[\zeta\left(F_{W|A=m} + b/\sqrt{T}\right) - \zeta\left(F_{W|A=m}\right)\right]',\ c_1',\ c_2 \right)'$$

for any $b \in \hat{\mathcal{D}}_T$, $c_1 \in \mathbb{R}^N$ and $c_2 \in \mathbb{R}$. For any $h \in \mathcal{B}_{[\underline{v},\overline{v}]}$, define $\zeta'(h) \equiv (\zeta_0'(h)$, $\zeta_1'(h)$, ..., $\zeta_{N-1}'(h))$ where for any $0 \leq a \leq N - 1$

$$\zeta_a'(h) \equiv \int_r^{\overline{v}} [\chi_{a,m}'(F_{W|A=m}(w)) - \chi_{a+1,m}'(F_{W|A=m}(w))]h(w)dw.$$

By construction, $\xi'(h)$ is a linear functional whose form depends implicitly on the actual conditional distribution of transaction prices $F_{W|A=m}(.)$ and $\phi_m^{-1}(.)$.

Proof of Proposition 3. Consider any sequence x_T such that (i) $x_T \equiv (x_{T,1}, x_{T,2}, x_{T,3}) \in \hat{\mathcal{D}}_T \otimes \mathbb{R}^{N+1}$ for all T; and (ii) $x_T \to x$ for some $x \equiv (x_1, x_2, x_3) \in \mathcal{B}_{[\underline{v},\overline{v}]} \otimes \mathbb{R}^{N+1}$. By definition, this means

$$g_T(x_T) = \left(\sqrt{T}\left[\zeta\left(F_{W|A=m,T}\right) - \zeta\left(F_{W|A=m}\right)\right]',\ (x_{T,2})',\ x_{T,3}\right)'$$

for a sequence $F_{W|A=m,T} \equiv F_{W|A=m} + T^{-\frac{1}{2}} x_{T,1} \in \hat{\mathcal{D}}_{[\underline{v},\overline{v}]}$ that satisfies $\sqrt{T}\left(F_{W|A=m,T} - F_{W|A=m}\right) \to F^*$ with $F^* = x_1 \in \mathcal{B}_{[\underline{v},\overline{v}]}$. It follows from Lemma B4 that under Assumptions 1 and 2,

$$\sqrt{T}\left[\zeta(F_{W|A=m,T}) - \zeta(F_{W|A=m})\right] \to \zeta'(F^*).$$

Hence it then follows that

$$g_T(x_T) \to (\ (\zeta'(x_1))',\ x_2',\ x_3\ )' \equiv g(x)$$

for any such sequence x_T. Next, let

$$X_T^* \equiv \begin{pmatrix} \sqrt{T}\left(\hat{F}_{W|A=m,T} - F_{W|A=m}\right) \\ \sqrt{T}\left(\hat{\rho}_T - \rho\right) \\ \sqrt{T}\left(\frac{1}{T}\sum_t \tilde{K}_t - \mu_K\right) \end{pmatrix};\ \text{and}\ X^* \equiv \begin{pmatrix} \mathbb{F}_{W|A=m} \\ \mathcal{N}_\rho \\ \mathcal{N}_\mu \end{pmatrix}.$$

By construction, the support of X_T^* is $\hat{\mathcal{D}}_T \otimes \mathbb{R}^{N+1}$ while the support of X^* is $\mathcal{B}_{[\underline{v},\overline{v}]} \otimes \mathbb{R}^{N+1}$. Under Assumptions 1 and 2, X_T^* converges weakly to X^* by Lemma B5 and the limiting stochastic process X^* is tight and measurable due to the Donsker property of the class of indicator functions of lower rectangles and equalities. Therefore it follows from the Extended Continuous Mapping Theorem (e.g. Theorem 1.11.1 in van der Vaart and Wellner (1996)) that

$$g_T(X_T^*) \equiv \begin{pmatrix} \sqrt{T}\left[\zeta(\hat{F}_{W|A=m,T}) - \zeta\left(F_{W|A=m}\right)\right] \\ \sqrt{T}\left(\hat{\rho}_T - \rho\right) \\ \sqrt{T}\left(\frac{1}{T}\sum_t \tilde{K}_t - \mu_K\right) \end{pmatrix} \rightsquigarrow g(X^*) \equiv \begin{pmatrix} \zeta'(\mathbb{F}_{W|A=m}) \\ \mathcal{N}_\rho \\ \mathcal{N}_\mu \end{pmatrix}.$$

An application of the Multivariate Delta Method then shows that $\sqrt{T}\left(\hat{\tau}_T - \tau_0\right) \rightsquigarrow \mathcal{N}_\tau \equiv \rho \cdot \zeta'(\mathbb{F}_{W|A=m}) + \zeta(F_{W|A=m}) \cdot \mathcal{N}_\rho - \mathcal{N}_\mu$, where "$\cdot$ " denotes the dot product between vectors. It then follows from Lemma 3.9.8. of van der Vaart and Wellner (1996) that $\mathcal{N}_\tau$ is a univariate normal with a zero mean and a finite variance. $\square$

C Bootstrap Inference: Consistency and Asymptotic Validity

Let $c_{1-\alpha/2}$ denote the actual $(1-\alpha/2)$-quantile of the limiting distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$. We estimate $c_{1-\alpha/2}$ using the following bootstrap procedure.

Step 1: Calculate $\hat{\tau}_T$ using the original estimation sample.

Step 2: Draw a bootstrap sample with size T from the original sample with replacement. Estimate τ_0 using this bootstrap sample and denote the estimate by $\hat{\tau}_{T,1}$.

Step 3: Repeat Step 2 for B times and denote the bootstrap estimates by $\{\hat{\tau}_{T,b}\}_{b \leq B}$. Find the $(1-\alpha)$-quantile of the empirical distribution of $\{\sqrt{T}|\hat{\tau}_{T,b} - \hat{\tau}_T|\}_{b \leq B}$. Denote it by $\hat{c}_{1-\alpha/2,T}$.

Let $G_T(.; F)$ and $G_\infty(.; F)$ denote respectively the finite sample distribution and the limiting distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ when the actual joint distribution of (W, A, K, ϵ) is given by a generic permissible distribution F. Let F_T and F_0 denote the empirical distribution of (W, A, K, ϵ) and its true distribution in the data-generating process respectively.

Lemma C6 *(Bootstrap Consistency) Suppose Assumptions 1 and 2 hold. Then for any $\varepsilon > 0$,*

$$\lim_{T\to\infty} \Pr\left(\sup_{\tau} |G_T(\tau; F_T) - G_\infty(\tau, F_0)| \ > \varepsilon\right) = 0$$

regardless of bidders' risk attitudes.

Proof of Lemma C6. By Proposition 3, $\sqrt{T}(\hat{\tau}_T - \tau_0)$ converges weakly to a zero-mean normal distribution whose variance depends on F_0. Across auctions in the data, $(W_t, A_t, K_t, \epsilon_t)$ are i.i.d. draws from F_0, and by the Uniform Law of Large Numbers, the empirical distribution of $(W_t, A_t, K_t, \epsilon_t)$ converges in probability to its population counterpart uniformly over the joint support. Also by Proposition 3, for any permissible joint distribution of (W, A, K, ϵ) in the data-generating process, the limiting distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ is continuous over $\mathbb{R}$.

Next, we will show that for any sequence of permissible distributions H_T that converges to H_0 uniformly as $T \to +\infty$, the distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ under H_T (denoted by $G_T(.; H_T)$) converges to $G_\infty(.; H_0)$ pointwise on the real line. Note for any $s \in \mathbb{R}$ and pair of permissible distributions H_1 and H_2,

$$|G_T(s; H_1) - G_\infty(s; H_2)| \leq |G_T(s; H_1) - G_\infty(s; H_1)| + |G_\infty(s; H_1) - G_\infty(s; H_2)| . \quad \text{(C20)}$$

For any s, the first term on the right-hand side of (C20) converges to 0 as $T \to +\infty$ by weak convergence of the test statistic under H_1.

Let $\|.\|_\infty$ denote the supremum norm. We now show that, for any $s \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\eta > 0$ (possibly depending on s, ε) such that for any pair H_1, H_2 with $\|H_1 - H_2\|_\infty \leq \eta$, $|G_\infty(s; H_1) - G_\infty(s; H_2)| < \varepsilon$. To see this, note by Proposition 3, for any generic distribution of (W, A, K, ϵ), denoted by H, that satisfies Assumptions 1 and 2, the limiting distribution $G_\infty(.; H)$ is a univariate normal with a zero mean and a variance that depends on the moments of (W, A, K, ϵ) under the distribution H (see the second and the third paragraph following Lemma B4 in Appendix B above). More specifically, the variance of the distribution $G_\infty(.; H)$ takes the form of $D^*\Sigma^* (D^*)'$, where $D^* \equiv (\rho', \zeta(F_{W|A=m})', -1) \in \mathbb{R}^{2N+1}$ and Σ^* is the covariance matrix for the random vector $(\zeta'(\mathbb{F}_{W|A=m}), \mathcal{N}_\rho', \mathcal{N}_\mu)'$. By Lemma B4, ζ (and therefore D^*) is continuous in H under the uniform (supremum) metric between distributions. By Lemma 3.9.8. in van der Vaart and Wellner (1996), Σ^* is a smooth functional of the covariance kernels between $\mathbb{F}_{W|A=m}$ and $(\mathcal{N}_\rho', \mathcal{N}_\mu)'$, which are continuous functions of H and its moments. Because these moments are continuous in H, the covariance matrix Σ^* is continuous in H. It then follows that the univariate normal distribution $G_\infty(s, H)$ is continuous in H at any $s \in \mathbb{R}$.

Now consider any H_T converging to H_0 under the supremum metric as $T \to +\infty$. The results above imply that for any $\varepsilon > 0$, $|G_T(s; H_T) - G_\infty(s; H_0)| \leq \varepsilon$ for T large enough (or equivalently for $\|H_T - H_0\|_\infty$ small enough). The lemma then follows from Beran and Ducharme (1991) (or Theorem 2.1 in Horowitz (2000)). $\square$

To simplify the notation, we suppress dependence of $\mathcal{N}_\tau$ on the actual joint distribution of $(W, A, \tilde{K})$ in the data for the rest of the proof. Since the limiting distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ is absolutely continuous with positive density almost surely over $\mathbb{R}$, an immediate corollary of the preceding lemmae is that our bootstrap estimator $\hat{c}_{1-\alpha/2,T}$ converges in probability to the actual $(1-\alpha/2)$-quantile of the limiting distribution (i.e. $c_{1-\alpha/2}$). Such consistency holds regardless of bidders' risk attitudes.

Proof of Proposition 4. Suppose $\tau_0 = \tau^* > 0$ in the data-generating process. By definition,

$$\Pr\left(\sqrt{T}\hat{\tau}_T \geq \hat{c}_{1-\alpha/2,T} \mid \tau_0 = \tau^*\right) = \Pr\left(\sqrt{T}(\hat{\tau}_T - \tau_0) - \hat{c}_{1-\alpha/2,T} \geq -\sqrt{T}\tau_0 \mid \tau_0 = \tau^*\right).$$

It follows from Proposition 3 that $\sqrt{T}(\hat{\tau}_T - \tau_0)$ converges in distribution to a univariate normal $\mathcal{N}_\tau$ with a zero mean and a finite variance. For any $\varepsilon \in (0,1)$, let $c_\varepsilon < +\infty$ denote the ε-quantile of $\mathcal{N}_\tau$. Since $\hat{c}_{1-\alpha/2,T} \xrightarrow{p} c_{1-\alpha/2}$ and $\tau_0 > 0$ under H_A, we have $\lim_{T\to+\infty} \Pr(\hat{c}_{1-\alpha/2,T} < c_\varepsilon + \sqrt{T}\tau_0 \mid \tau_0 = \tau^*) \to 1$ for any $\varepsilon \in (0,1)$. Hence for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
& \Pr\left(\sqrt{T}\hat{\tau}_T \geq \hat{c}_{1-\alpha/2,T} \mid \tau_0 = \tau^*\right) \\
\geq\ & \Pr\left(\sqrt{T}\hat{\tau}_T \geq \hat{c}_{1-\alpha/2,T} \textit{ and } \hat{c}_{1-\alpha/2,T} < c_\varepsilon + \sqrt{T}\tau_0 \mid \tau_0 = \tau^*\right) \\
>\ & \Pr\left(\sqrt{T}\left(\hat{\tau}_T - \tau_0\right) \geq c_\varepsilon \textit{ and } \hat{c}_{1-\alpha/2,T} < c_\varepsilon + \sqrt{T}\tau_0 \mid \tau_0 = \tau^*\right) \\
\to\ & \lim_{T\to\infty} \Pr\left(\sqrt{T}\left(\hat{\tau}_T - \tau_0\right) \geq c_\varepsilon \mid \tau_0 = \tau^*\right) = 1 - \varepsilon
\end{aligned}$$

as $T \to +\infty$. This proves the consistency of our test under any fixed alternative of risk-averse bidders ($H_A : \tau_0 = \tau^* > 0$). A symmetric argument shows

$$\lim_{T\to+\infty} \Pr\left(\sqrt{T}\hat{\tau}_T \leq -\hat{c}_{1-\alpha/2,T} \mid \tau_0 = \tau^*\right) = 1$$

for any $\tau^* < 0$ (bidders are risk-loving). If bidders are risk-neutral with $\tau_0 = 0$,

$$\begin{aligned}
& \Pr\left(-\hat{c}_{1-\alpha/2,T} \leq \sqrt{T}\hat{\tau}_T \leq \hat{c}_{1-\alpha/2,T} \mid \tau_0 = 0\right) \\
=\ & \Pr\left(\sqrt{T}\left(\hat{\tau}_T - \tau_0\right) + \hat{c}_{1-\alpha/2,T} \geq 0 \textit{ and } \sqrt{T}\left(\hat{\tau}_T - \tau_0\right) - \hat{c}_{1-\alpha/2,T} \leq 0 \mid \tau_0 = 0\right) \\
\longrightarrow\ & \Pr(-c_{1-\alpha/2} \leq \mathcal{N}_\tau \leq c_{1-\alpha/2}) = 1 - \alpha \text{ as } T \to +\infty,
\end{aligned}$$

where the second equality follows from that $\sqrt{T}\left(\hat{\tau}_T - \tau_0\right) \xrightarrow{d} \mathcal{N}_\tau$ and $\hat{c}_{1-\alpha/2,T} \xrightarrow{p} c_{1-\alpha/2}$ ($1-\alpha/2$ quantile of the zero-mean normal variable $\mathcal{N}_\tau$) and an application of the Slutsky's Theorem. $\square$

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