Web Appendix to "Specification and Negotiation in Incomplete Contracts"

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A. Non-identification without Normalization

For a single-agent model with a non-additive monotone outcome function, Matzkin (2003) showed some scale normalization of unobserved shocks is necessary for nonparametric identification.[1] In comparison, we show that normalizing the distribution of the buyer's private signals is necessary for identifying our game-theoretic model, where the vector of outcomes (V, D, Y, X, X^*) reported are rationalized by strategic interaction between contractors and the buyer in monotone psPBE. To do so, we need to use a two-step approach that is qualitatively different from Matzkin (2003). We first characterize how players' equilibrium strategies vary with model parameters. Then we show how distinct sets of model parameters could generate the same distribution of outcome (V, Y, D, X, X^*) in equilibrium.

We say two models are *observationally equivalent* if they imply the same distribution of V, Y, D, X and X^* in symmetric psPBE. Suppose the actual data-generating process is $\theta \equiv \{\gamma, \pi, a, F_{C|X}, F_{X^*,\tilde{X}}\}$. Consider an alternative model $\theta_0 \equiv \{\gamma_0, \pi_0, a_0, H_{C|X}, H_{X^*,\tilde{X}}\}$ such that (i) $a = a_0$, $\pi = \pi_0$ and $\gamma = \gamma_0$; (ii) $F_{C|X=x} = H_{C|X=x}$ for all $x \in \mathcal{X}$; and (iii) $F_{X^*,\tilde{X}}(x, \tilde{x}) = H_{X^*,\tilde{X}}(x, h(\tilde{x}))$ for all $x, \tilde{x}$ and some increasing and differentiable function $h : \widetilde{\mathcal{X}} \to \mathcal{T} \subset \mathbb{R}$. In other words, all structural elements in θ and θ_0 are identical, except that the procurer's private signal in θ_0 is an increasing and differentiable transformation of that in θ. Note the support of the private signal in θ_0 (denoted $\mathcal{T}$) is allowed to differ from that in θ.

Proposition A1. *Suppose θ and θ_0 are two data-generating processes satisfying (i)-(iii) in the preceding paragraph, then θ and θ_0 are observationally equivalent.*

[1] Matzkin (2003) considered a nonparametric model $Y = m(X, \epsilon)$ where m is monotone in ϵ, and ϵ is independent from X. Lemma 1 in Matzkin (2003) established that, without further restrictions on m, the model is observationally equivalent to another model $Y = \widetilde{m}(X, \tilde{\epsilon})$ where $\tilde{\epsilon}$ is any monotone transformation of ϵ.

Proof. The first step is to relate the equilibrium strategy in θ to that in θ_0. Let $\alpha(\cdot;\theta):\widetilde{\mathcal{X}}\to\mathcal{X}$ denote the procurer strategy under θ in a symmetric monotone psPBE; and likewise define $\alpha(\cdot;\theta_0)$. Let $\alpha,\dot{\alpha}$ be shorthand for $\alpha(\tilde{x})$ and $\alpha'(\tilde{x})$ respectively. Recall that in equilibrium the procurer's strategy $\alpha(\cdot;\theta)$ solves the ordinary differential equation (ODE):

$$\dot{\alpha}=\frac{\gamma\Lambda_2(\alpha,\tilde{x};\theta)}{\sigma'(\alpha;\theta)-\pi'(\alpha)-\Lambda_1(\alpha,\tilde{x};\theta)}\equiv\varpi(\alpha,\tilde{x}) \tag{1}$$

for an initial condition $\alpha(\tilde{x}_l;\theta)=x_l^o$, where μ,δ,σ are defined above. We highlight the dependence of $\Lambda_1,\Lambda_2,\sigma$ on θ throughout this subsection. While Λ_1 and Λ_2 depend on $\{\gamma,\pi,a,F_{X^*,\tilde{X}}\}$, $\sigma:\mathcal{X}\to\mathbb{R}$ only depends on $F_{C|X}$ in θ. Likewise, $\alpha(\cdot;\theta_0)$ solves an ODE similar to (1), only with θ therein replaced by θ_0:

$$\dot{\alpha}_0=\frac{\gamma\Lambda_2(\alpha_0,\tilde{x};\theta_0)}{\sigma'(\alpha_0;\theta)-\pi'(\alpha_0)-\Lambda_1(\alpha_0,\tilde{x};\theta_0)} \tag{2}$$

and a corresponding initial condition. By construction, $F_{C|X}=H_{C|X}$ and $F_{X^*|\tilde{X}=\tilde{x}}=H_{X^*|\tilde{X}=h(\tilde{x})}$ for all $\tilde{x}\in\mathcal{X}$. Hence $\sigma'(\cdot;\theta)=\sigma'(\cdot;\theta_0)$ and

$$\Lambda_k(\cdot,h^{-1}(t);\theta)=\Lambda_k(\cdot,t;\theta_0),k=1,2 \tag{3}$$

for any $t\in\mathcal{T}$. Substitute (3) into the ODE for θ_0 in (2) gives:

$$\dot{\alpha}_0=\frac{\gamma\Lambda_2(\alpha_0,h^{-1}(t);\theta)}{\sigma'(\alpha_0;\theta)-\pi'(\alpha_0)-\Lambda_1(\alpha_0,h^{-1}(t);\theta)}\left(\frac{dh^{-1}(t)}{dt}\right) \tag{4}$$

for any $t\in\mathcal{T}$. To establish the relation between $\alpha(\cdot;\theta)$ and $\alpha(\cdot;\theta_0)$, we need to show the following auxiliary lemma.

Lemma A1. Let $\zeta:\mathcal{T}\to\mathcal{X}$ be an increasing differentiable function, where $\mathcal{T}$, $\mathcal{X}$ are open sets on $\mathbb{R}$. If $\alpha(\tilde{x})$ solves the ODE "$\dot{\alpha}=\varpi(\alpha,\tilde{x})$, $\tilde{x}\in\mathcal{X}$" with an initial condition $\alpha(\tilde{x}_l)=x_l^o$, then $\alpha_0(t)\equiv\alpha(\zeta(t))$ solves "$\dot{\alpha}_0=\varpi(\alpha_0,\zeta(t))\zeta'(t)$, $t\in\mathcal{T}$" with an initial condition $\alpha_0(\zeta^{-1}(\tilde{x}_l))=x_l^o$.

Proof of Lemma A1. By the definition of $\alpha(\tilde{x})$ as a solution to the first ODE, we have $\alpha'(\tilde{x})=\varpi(\alpha(\tilde{x}),\tilde{x})$ for all $\tilde{x}\in\mathcal{X}$. If $\alpha_0(t)\equiv\alpha(\zeta(t))$ for all $t\in\mathcal{T}$, then $\alpha_0'(t)=\alpha'(\zeta(t))\zeta'(t)$ for all $t\in\mathcal{T}$. Combining these two equalities, we have $\alpha_0'(t)=\varpi(\alpha(\zeta(t)),\zeta(t))\zeta'(t)$ for all $t\in\mathcal{T}$. With $\alpha_0(t)\equiv\alpha(\zeta(t))$, this equation can be written as $\alpha_0'(t)=\varpi(\alpha_0(t),\zeta(t))\zeta'(t)$ for all $t\in\mathcal{T}$. Finally, note $\alpha(\tilde{x}_l)=x_l^o$ and $\alpha_0(t)=\alpha(\zeta(t))$ implies $\alpha_0(\zeta^{-1}(\tilde{x}_l))=\alpha(\tilde{x}_l)=x_l^o$. This proves Lemma A1. $\square$

An application of Lemma A1 to (4) with $\zeta\equiv h^{-1}$ and $\varpi(\alpha,\tilde{x})$ defined on the r.h.s. of (1) implies the procurer equilibrium strategies under θ_0 and θ are related as $\alpha(t;\theta_0)=\alpha(h^{-1}(t);\theta)$ for all $t\in\mathcal{T}$.

Next, note that the joint distribution of (X^*,X) evaluated at (x^*,x) according to θ is $F_{X^*,\tilde{X}}(x^*,\alpha^{-1}(x;\theta))$ while that under θ_0 is $H_{X^*,\tilde{X}}(x^*,\alpha^{-1}(x;\theta_0))$. The relation between $\alpha(\cdot;\theta)$ and $\alpha(\cdot;\theta_0)$ implies that $\alpha^{-1}(x;\theta_0)=h(\alpha^{-1}(x;\theta))$ for all $x\in\mathcal{X}$. Hence

$$H_{X^*,\tilde{X}}(x^*,\alpha^{-1}(x;\theta_0))=H_{X^*,\tilde{X}}(x^*,h(\alpha^{-1}(x;\theta)))=F_{X^*,\tilde{X}}(x^*,\alpha^{-1}(x;\theta))$$

where the second equality is due to the specified relation between $F_{X^*,\tilde{X}}$ and $H_{X^*,\tilde{X}}$. Therefore the two joint distributions of (X^*, X) in equilibrium under θ and θ_0 are identical. Let $\phi(x, x^*) \equiv \pi(x^*) - \pi(x)$ and $\phi_0(x, x^*) = \pi_0(x^*) - \pi_0(x)$. Then that $a = a_0$, $\pi = \pi_0$ and $\gamma = \gamma_0$ in θ and θ_0 imply $\gamma\phi + (1-\gamma)a = \gamma_0\phi_0 + (1-\gamma)a_0$ for all x, x^*. Furthermore, a pair (X, X^*) leads to positive net surplus $\phi - a$ under θ if and only if it does so under θ_0. Hence the two models θ and θ_0 imply the same probability of $D = 1$ given (X, X^*), and the same distribution of Y given $D = 1$ and (X, X^*) in equilibrium.

Finally, the distribution of prices quoted by contractors at equilibrium under θ is determined by the distribution of costs $F_{C|X}$ as well as the hold-up $\delta(X; \alpha(\cdot; \theta)) \equiv \mathbb{E}[\gamma s_+(X, X^*) \mid X; \alpha(\cdot; \theta)]$. By construction, $F_{C|X} = H_{C|X}$ and $\gamma(\phi - a) = \gamma_0(\phi_0 - a_0)$. Furthermore, the argument above already shows that the joint distribution of (X, X^*) in psPBE is identical under θ and θ_0. It then follows that θ and θ_0 can generate the same joint distribution of V, Y, D, X and X^* in psPBE. □

B. Identification in the Parametric Model

The model in Section 5 accommodates both contract and contractor heterogeneity and includes additive structural errors ε and η. The following proposition establishes the identification of this parametric model under mild support conditions of the explanatory variables. We assume structural parameters are non-zero (that is, $\theta_r \neq 0$ for $r = 1, 2, ..., 6$, $\sigma \neq 0$ and $\lambda \neq 0$.)

Proposition B1. *In the model (10)-(11) with the specification (12)-(13), the structural parameters* $\{\theta_j\}_{j=1,...,6}, \varrho, \rho$ *and* σ^2 *are identified if the support of* $[(x^* - x), (x^* - x) \times job, x^{*2} - x^2, (x^* - x)^2, w]$ *has full rank.*

We now sketch a proof of this identification result. To simplify presentation, we first condition on the other explanatory variables in w (other than job) and suppress them in the notation below. Let γ_k be shorthand for the bargaining power when $job = k \in \{0, 1\}$. To begin with, expand the latent variable $\gamma_k\phi + (1-\gamma_k)a + \varepsilon$ in the outcome equation (10) into:

$$
\begin{aligned}
& \gamma_k[\theta_1(x^* - x) + \theta_2(x^* - x)k + \theta_3(x^{*2} - x^2)] \\
& +(1-\gamma_k)[\theta_0 + \theta_4(x^* - x) + \theta_5(x^* - x)k + \theta_6(x^* - x)^2] + \varepsilon \\
= \; & \beta_{k,0} + \beta_{k,1}(x^* - x) + \beta_{k,2}x^2 + \beta_{k,3}x^{*2} + \beta_{k,4}(x^* - x)k + \beta_{k,5}x^*x + \varepsilon
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_{k,0} & \equiv (1-\gamma_k)\theta_0;\ \beta_{k,1} \equiv \gamma_k\theta_1 + (1-\gamma_k)\theta_4;\ \beta_{k,2} \equiv [(1-\gamma_k)\theta_6 - \gamma_k\theta_3]; \\
\beta_{k,3} & \equiv \gamma_k\theta_3 + (1-\gamma_k)\theta_6;\ \beta_{k,4} \equiv [\gamma_k\theta_2 + (1-\gamma_k)\theta_5];\ \beta_{k,5} \equiv -2(1-\gamma_k)\theta_6
\end{aligned}
$$

for $k \in \{0, 1\}$.

Also we can write the latent index in the selection equation (11) as

$$\begin{aligned} & \theta_1(x^* - x) + \theta_2(x^* - x)k + \theta_3(x^{*2} - x^2) - [\theta_0 + \theta_4(x^* - x) + \theta_5(x^* - x)k + \theta_6(x^* - x)^2] \\ = \ & \tilde{\beta}_0 + \tilde{\beta}_1(x^* - x) + \tilde{\beta}_2(x^* - x)job + \tilde{\beta}_3(x^{*2} - x^2) + \tilde{\beta}_4(x^* - x)^2 + \eta \end{aligned}$$

where

$$\tilde{\beta}_0 \equiv -\theta_0; \ \tilde{\beta}_1 \equiv \theta_1 - \theta_4; \ \tilde{\beta}_2 \equiv \theta_2 - \theta_5; \ \tilde{\beta}_3 \equiv \theta_3; \ \tilde{\beta}_4 \equiv -\theta_6.$$

Recall the coefficients in a probit model are identified up to scale (because of the normalization on the standard deviation in probit), provided the support of the explanatory variables has full rank. Thus, in our model the variation in x and x^* on the equilibrium path helps us recover $\beta_j^* \equiv \tilde{\beta}_j/\sigma_\eta$ for $j = 0, 1, 2, 3, 4\}$ (and therefore $(\theta_1 - \theta_4)/\sigma_\eta$, $(\theta_2 - \theta_5)/\sigma_\eta$ and θ_0/σ_η, θ_3/σ_η, θ_6/σ_η) from the conditional probability of incompleteness given x, x^* and job.

Next, note the expectation of the negotiated transfer conditional on $x, x^*, job = 1, d = 1$ is

$$\beta_{1,0} + (\beta_{1,1} + \beta_{1,4})(x^* - x) + \beta_{1,2}x^2 + \beta_{1,3}x^{*2} + \beta_{1,5}x^*x + E[\varepsilon \mid x, x^*, job = 1, d = 1] \quad (5)$$

where by the bivariate normality of (ε, η), the last conditional expectation is the product of a constant coefficient and the inverse mill's ratio evaluated at

$$\beta_0^* + (\beta_1^* + \beta_2^*)(x^* - x) + \beta_3^*(x^{*2} - x^2) + \beta_4^*(x^* - x)^2$$

where $\{\beta_j^* : j = 0, 1, 2, 3, 4\}$ are identified from the selection equation (11). Thus $\beta_{1,0}, \beta_{1,1} + \beta_{1,4}, \beta_{1,2}, \beta_{1,3}, \beta_{1,5}$ are identified using (5) under a typical rank condition.[2]

It then follows that $\theta_6/\theta_3 = -\beta_4^*/\beta_3^* \equiv r^*$ and

$$\begin{aligned} \frac{\beta_{1,2}}{\beta_{1,3}} &= \frac{(1-\gamma_1)\theta_6 - \gamma_1\theta_3}{(1-\gamma_1)\theta_6 + \gamma_1\theta_3} = \frac{r^*(1-\gamma_1) - \gamma_1}{r^*(1-\gamma_1) + \gamma_1} \\ \Rightarrow \gamma_1 &= \frac{r^*(\beta_{1,3} - \beta_{1,2})}{r^*(\beta_{1,3} - \beta_{1,2}) + \beta_{1,2} + \beta_{1,3}}. \end{aligned} \quad (6)$$

Once γ_1 is identified, so is

$$\begin{aligned} \theta_0 &= \beta_{1,0}/(1-\gamma_1); \ \sigma_\eta = -\theta_0/\beta_0^* = \frac{\beta_{1,0}}{(1-\gamma_1)\beta_0^*}; \\ \theta_3 &= \sigma_\eta\beta_3^* = \frac{\beta_{1,0}\beta_3^*}{(1-\gamma_1)\beta_0^*}; \ \theta_6 = -\sigma_\eta\beta_4^* = -\frac{\beta_{1,0}\beta_4^*}{(1-\gamma_1)\beta_0^*}. \end{aligned}$$

[2] The rank condition requires the support of $x^* - x, x^2, x^{*2}, x^*x$ and the mill's ratio to have full rank. This condition can be satisfied under mild conditions due to the nonlinearity of the mill's ratio (even when the selection equation does not involve any exogenous variable that is excluded from the outcome equation). See the last but one paragraph on page 806 in Wooldridge (2010) (which starts with "As a technical point, we do not ...") for more detailed discussion.

Likewise, we can identify $\beta_{0,0}, \beta_{0,1}, \beta_{0,2}, \beta_{0,3}, \beta_{0,5}$ from the conditional expectation of transfer given $x, x^*, job = 0, d = 1$ which equals

$$\beta_{0,0} + \beta_{0,1}(x^* - x) + \beta_{0,2}x^2 + \beta_{0,3}x^{*2} + \beta_{0,5}x^*x + E[\varepsilon \mid x, x^*, job = 0, d = 1].$$

Then by symmetric argument as in (6), one can identify γ_0 using $\beta_{0,2}, \beta_{0,3}$ in place of $\beta_{1,2}, \beta_{1,3}$.

It remains to show $\theta_1, \theta_2, \theta_4, \theta_5$ are identified. Note that we have already recovered: (a) $\gamma_1\theta_1 + (1-\gamma_1)\theta_4 + [\gamma_1\theta_2 + (1-\gamma_1)\theta_5]$ from the outcome equation conditional on $job = 1$; (b) $\gamma_0\theta_1 + (1-\gamma_0)\theta_4$ from the outcome equation conditional on $job = 0$; and (c) $\theta_1 - \theta_4$ and $\theta_2 - \theta_5$ from the selection equation (11) using σ_η, which is already identified. Thus we can construct a linear system of four equations and four unknowns

$$\begin{pmatrix} \gamma_0 & 1-\gamma_0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ \gamma_1 & 1-\gamma_1 & \gamma_1 & 1-\gamma_1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_4 \\ \theta_2 \\ \theta_5 \end{pmatrix} = r.h.s.$$

where the right-hand side consists of quantities that are already identified. The coefficient matrix on the l.h.s. is non-singular for all $\gamma_0, \gamma_1 \in [0, 1]$. It then follows $\theta_1, \theta_2, \theta_4, \theta_5$ are all identified. Finally, recall that the identification result above was shown by conditioning on w, which means the bargaining power, as a function of characteristics w and the job type, is identified for all values w and job. The identification of ϱ then follows from the parametric form of $\gamma(w, job; \varrho)$ and the full-rankness of the support of w.

References

Matzkin, R. L. (2003): "Nonparametric estimation of nonadditive random functions," *Econometrica*, 71(5), 1339–1375.